# Variance reduction

## Antithetic variables

The use of antithetic variables produces a sequence of realisations $\left(Y_1, \tilde{Y}_1\right), \left(Y_2, \tilde{Y}_2\right), \ldots, \left(Y_n, \tilde{Y}_n\right)$ with

- the pairs $\left(Y_1, \tilde{Y}_1\right), \left(Y_2, \tilde{Y}_2\right), \ldots, \left(Y_n, \tilde{Y}_n\right)$ are i.i.d.
- for each $i$, $Y_i$ and $\tilde{Y}_i$ have the same distribution, though they are not independent.

The antithetic variables estimator is the average of all $2n$ observations

$$\hat{Y}_{\text{AV}} = \frac{1}{2\,n}\left(\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \tilde{Y}_i\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{Y_i + \tilde{Y}_i}{2}\right)$$

The variance of the AV estimator is

$$V[\hat{Y}_{\text{AV}}] = \frac{1}{n^2}\sum_{i=1}^{n} V\left[\frac{Y_i + \tilde{Y}_i}{2}\right] = \frac{1}{n} V\left[\frac{Y_i + \tilde{Y}_i}{2}\right]$$

Now

$$V\left[Y_i + \tilde{Y}_i\right] = V[Y_i] + V[\tilde{Y}_i] + 2\,\text{Cov}\left[Y_i + \tilde{Y}_i\right] = 2\,V[Y_i] + 2\,\text{Cov}\left[Y_i, \tilde{Y}_i\right]$$

since $V[\tilde{Y}_i] = V[Y_i]$. Therefore

$$V\left[\frac{Y_i + \tilde{Y}_i}{2}\right] = \frac{1}{4}\left(2\,V[Y_i] + 2\,\text{Cov}\left[Y_i, \tilde{Y}_i\right]\right) = \frac{V[Y_i] + \text{Cov}\left[Y_i, \tilde{Y}_i\right]}{2}$$

and

$$V[\hat{Y}_{\text{AV}}] = \frac{V[Y_i] + \text{Cov}\left[Y_i, \tilde{Y}_i\right]}{2\,n}$$

Conversely, the variance of the estimator with $2n$ *independent* observations is

$$V[\hat{Y}_{2\,n}] = \frac{V[Y_i]}{2\,n}$$

Therefore, the AV estimator has lower variance provided $\text{Cov}\left[Y_i, \tilde{Y}_i\right] < 0$, that is

$$\hat{Y}_{\text{AV}} < \hat{Y}_{2\,n} \Longleftrightarrow \text{Cov}\left[Y_i, \tilde{Y}_i\right] < 0$$

A *sufficient* condition for variance reduction with uniform or normal random variables is that the outcome function is monotonic, that is

$$X_i < X_j \Longrightarrow Y_i < Y_j \quad \text{or} \quad Y_i > Y_j$$

where $X$ is the underlying random variable (e.g. $X \sim U$ or $Z$ ).

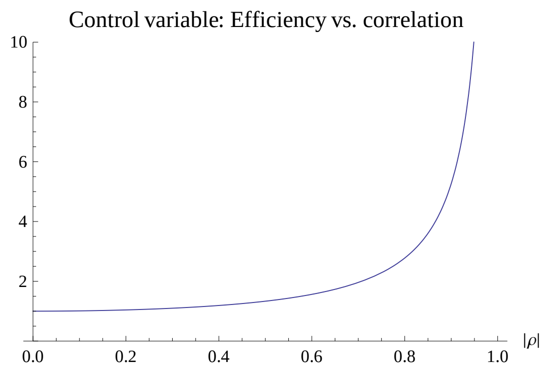
Since producing $n$ antithetic pairs is normally faster than producing $2n$ independent replications, antithetic variables should be used whenever possible.

## Control variables

With $\alpha$ selected optimally, that is $\alpha^* = \text{Cov}[Y, \text{CV}]/\text{Var}[\text{CV}]$, the ratio of the variances of the control variable estimator to the standard estimator is

$$\frac{\text{Var}\left[\overline{Y} - \alpha^*\left(\overline{\text{CV}} - E[\text{CV}]\right)\right]}{\text{Var}[\overline{Y}]} = 1 - \rho^2_{Y,\text{CV}}$$

- The effectiveness of the control variable method depends solely upon the degree of correlation between target variable and the control variable (the sign is absorbed into $\alpha$).
- If computation of the control variable does not add significantly to the time required, $\frac{1}{1-\rho^2}$ approximately measures the efficiency improvement of using the control variable.




The degree of improvement drops sharply as correlation declines. A correlation of 0.95 produces a 10-fold increase in efficiency, a correlation of 0.9 produces a 5-fold increase, and a correlation of 0.7 produces only a 2-fold increase.

- In practice, the population moments are unknown and $\alpha^*$ is estimated from the sample. This introduces some bias in finite samples. This bias can be eliminated by estimating $\alpha$ on a different random sample. Typically, the bias is so small that this is not worthwhile.
- When $\alpha^*$ is estimated from the sample, conventional confidence intervals are valid asymptotically (Glasserman 2004: 195).

Control variables are especially valuable in the simulation of Asian and spread options.

# Moment matching

Disadvantage: Confidence intervals are difficult to obtain, since the realizations are no longer independent.

# Importance sampling

# Stratified sampling

# Latin hypercube sampling