Sensitivity analysis

Simulating the Greeks

Meet the Greeks

The value of a derivative on a single underlying asset depends upon the current asset price S and its volatility σ, the risk-free interest rate r, and the time to maturity t. That is, $V = f(S, r, \sigma, t)$. (It also depends upon constants like the strike price K.) Taking a Taylor series expansion, the change in value over a small time period can be approximated by

$$dV \approx \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \sigma} d\sigma + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2$$
$$+ \text{other second order terms}$$
$$+ \text{higher order terms}$$

The partial derivatives in this expansion are known collectively as "the Greeks". They measure the sensitivity of a portfolio to changes in the underlying parameters. Specifically

$\Delta = \frac{\partial f}{\partial S}$ **Delta** measures the sensitivity of the portfolio value to changes in the price of the underlying

$\rho = \frac{\partial f}{\partial r}$ **Rho** measures the sensitivity of the portfolio value to changes in the interest rate

$v = \frac{\partial f}{\partial \sigma}$ **Vega** measures the sensitivity of the portfolio value to changes in the volatility of the underlying

$\Theta = \frac{\partial f}{\partial t}$ **Theta** measures the sensitivity of the portfolio value to the passage of time

$\Gamma = \frac{\partial^2 f}{\partial S^2} = \frac{\partial \Delta}{\partial S}$ **Gamma** measures the sensitivity of delta to changes in the price of the underlying, or the curvature of the $S - V$ curve.

Substituting in (1), the change in value of the portfolio can be approximated by

$$dV \approx \Delta\, dS + \rho\, \mathrm{dr} + v\, d\sigma + \Theta\, \mathrm{dt} + \frac{1}{2} \Gamma\, \mathrm{dS}^2$$

Because differentiation is a *linear operator*, the hedge parameters of a portfolio are equal to a weighted average of the hedge parameters of its components. In particular, the hedge parameters of a short position are the negative of the hedge parameters of a long position. Consequently, (2) applies equally to a portfolio as to an individual asset. The sensitivity of a portfolio to the risk factors (S, r, σ) can be altered by changing the composition of the portfolio. It can be reduced by adding assets with offsetting parameters.

The Greeks are not independent. Any derivative (or portfolio of derivatives) $V = f(S, r, \sigma, t)$ must satisfy the Black-Scholes differential equation

$$\frac{\partial f}{\partial t} + r S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r V$$

Substituting

$$\frac{\partial \Pi}{\partial t} = \Theta \qquad \frac{\partial \Pi}{\partial S} = \Delta \qquad \frac{\partial^2 \Pi}{\partial S^2} = \Gamma$$

it follows that the Greeks must satisfy the following relationship

$$\Theta + r\,S\,\Delta + \frac{1}{2}\sigma^2 S^2\,\Gamma = r\,V$$

Computing the Greeks

Formulae for vanilla European options

The Greeks of vanilla European options have straightforward formulae, which can be derived from the Black-Scholes formula. The Black-Scholes formulae for European options are

$$c = S\,e^{-q\,T}\,N(d_1) - K\,e^{-r\,T}\,N(d_2)$$

$$p = K\,e^{-r\,T}\,N(-d_2) - S\,e^{-q\,T}\,N(-d_1)$$

where

$$d_1 = \frac{\ln(S/K) + \left(r - q + \sigma^2\,T\right)/2}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln(S/K) + \left(r - q - \sigma^2\,T\right)/2}{\sigma\sqrt{T}} = d_1 - s$$

The partial derivatives ("the Greeks") are

	Call	**Put**
Delta	$e^{-q\,T}\,N(d_1)$	$e^{-q\,T}(N(d_1) - 1)$
Gamma	$\dfrac{e^{-q\,T}\,N'(d_1)}{S\,\sigma\sqrt{T}}$	$\dfrac{e^{-q\,T}\,N'(d_1)}{S\,\sigma\sqrt{T}}$
Rho	$K\,T\,e^{-r\,T}\,N(d_2)$	$-K\,T\,e^{-r\,T}\,N(-d_2)$
Vega	$e^{-q\,T}\,S\sqrt{T}\,N'(d_1)$	$e^{-q\,T}\,S\sqrt{T}\,N'(d_1)$
Theta	$-\dfrac{e^{-q\,T}\,S\,\sigma\,N'(d_1)}{2\sqrt{T}} + q\,e^{-q\,T}\,S\,N(d_1) - r\,K\,e^{-r\,T}\,N(d_2)$	$-\dfrac{e^{-q\,T}\,S\,\sigma\,N'(d_1)}{2\sqrt{T}} - q\,e^{-q\,T}\,S\,N(-d_1) - r\,K\,e^{-r\,T}\,N(-d_2)$

As an example of the derivation, for a call option

$$\Gamma = \frac{\partial\Delta}{\partial S} = e^{-q\,T}\,N'(d_1)\,\frac{\partial d_1}{\partial S} = e^{-q\,T}\,N'(d_1)\,\frac{1}{S\,\sigma\sqrt{T}}$$

Calculating vega from the Black-Scholes formula is an approximation, since the formula is derived under the assumption that volatility is constant. Fortunately, it can be shown that it is a good approximation to the vega calculated from a stochastic volatility model (Hull 2003: 318).

Some exotic options (e.g. barrier options) have analogous formulae. However, for most exotic options, the Greeks must be estimated by numerical techniques. Since these are the type of options for which institutions require such information, this motivates are interest in the accurate computation of option values and sensitivities.

Numerical differentiation in general

In principle, the Greeks for general options can be estimated by numerical differentiation. For example Delta, which measures the sensitivity of the option value to changes in the price of the underlying, is defined as

$$\Delta = \frac{\partial V(S)}{\partial S} = \lim_{dS \to 0} \frac{V(S+dS) - V(S)}{dS}$$

An obvious method to evaluate Δ is to compute

$$\Delta \approx \frac{V(S + \mathrm{dS}) - V(S)}{\mathrm{dS}}$$

for small dS. This is known as the *forward difference*. A better alternative (though more costly to compute) is

$$\Delta \approx \frac{V(S+dS) - V(S - dS)}{2\,dS}$$

which is known as the *central difference*. The other first-order Greeks (rho, theta and vega) can be estimated similarly.

Gamma is the derivative of delta, or the second derivative of $V(S)$. Using central differences, gamma can be estimated by

$$\Gamma \approx \frac{\Delta\left(S + \frac{1}{2}\,dS\right) - \Delta\left(S - \frac{1}{2}\,dS\right)}{dS} = \frac{\frac{V(S+dS)-V(S)}{dS} - \frac{V(S)-V(S-dS)}{dS}}{dS} = \frac{\frac{V(S+dS)-V(S)}{dS} - \frac{V(S)-V(S-dS)}{dS}}{dS}$$

$$= \frac{V(S+dS) - 2\,V(S) + V(S - dS)}{dS^2}$$

- **Simulating the Greeks**

Finite differences

The delta of a derivative is

$$\Delta = \frac{\partial V}{\partial S_0}$$

With simulation, this can be estimated by the average of the finite-differences over a sample n of replications.

$$\tilde{\Delta} = \frac{1}{n} \sum \frac{\tilde{V}(S_0 + dS) - \tilde{V}(S_0)}{\mathrm{dS}}$$

If we use different random samples for estimating $\tilde{V}(S_0)$ and $\tilde{V}(S_0 + dS)$, the variance of $\tilde{\Delta}$ becomes very large as dS becomes small.

A better estimate of Δ is generally obtained by using the same random numbers in estimating both $\tilde{V}(S_0)$ and $\tilde{V}(S_0 + dS)$. The variance of $\tilde{\Delta}$ is

$$\mathrm{Var}[\tilde{\Delta}] = \frac{1}{dS^2}\left(\mathrm{Var}\left[\tilde{V}(S_0)\right] + \mathrm{Var}\left[\tilde{V}(S_0 + dS)\right] - 2\,\mathrm{Cov}\left[\tilde{V}(S_0),\ \tilde{V}(S_0 + dS)\right]\right)$$

Provided that $\tilde{V}(S_0)$ and $\tilde{V}(S_0 + dS)$ are positively correlated, the estimate obtained from common random numbers will have a lower variance.

Even with common random numbers, the finite difference estimator is biased. In general, the bias increases with dS, while the variance decreases with dS. Consequently, choice of the optimal value of dS involves a tradeoff between bias and variance. Except in the simplest cases, it is not possible to determine the optimal tradeoff analytically. The alternative techniques, the pathwise derivative and the likelihood ratio method are both unbiased, though they do not necessarily offer lower variance.

Pathwise derivative

The pathwise derivative decomposes the derivative into two components.

$$\frac{\partial \tilde{V}}{\partial S_0} = \frac{\partial \tilde{V}}{\partial S_T} \frac{\partial S_T}{\partial S_0}$$

Assuming geometric Brownian motion and the risk-neutral distribution

$$S_T = S_0\, e^{\left(r-q-\frac{\sigma^2}{2}\right)T + \sigma \sqrt{T}\, Z} \tag{1}$$

so that

$$\frac{\partial S_T}{\partial S_0} = e^{\left(r-q-\frac{\sigma^2}{2}\right)T + \sigma \sqrt{T}\, Z} = \frac{S_T}{S_0}$$

The discounted payoff of a vanilla European call option is $e^{-r\,T} \max(S_T - K,\ 0)$

$$\frac{\partial \tilde{V}}{\partial S_T} = e^{-r\,T} \frac{\mathrm{d}}{\mathrm{d}\, S_0} \max(S_T - K,\, 0) = \begin{pmatrix} e^{-r\,T}, & S_T > K \\ 0, & S_T < K \end{pmatrix} \tag{2}$$

If the option is in-the-money, a small increase dS_T in the terminal price increases the discounted payoff by $e^{-r\,T}\, dS_T$. If the option is out-of-the money, a small increase dS_T in the terminal price leaves the payout unchanged. The function is not differentiable at $S_T = K$, but this is a zero-probability event.

Combining the two factors, *the pathwise estimator of delta* for a vanilla European call option is

$$\frac{\partial \tilde{V}}{\partial S_0} = \frac{\partial \tilde{V}}{\partial S_T} \frac{\partial S_T}{\partial S_0} = e^{-r\,T} \frac{S_T}{S_0} \boldsymbol{I}\{S(T) > K\}$$

where $I\{S(T) > K\}$ equals one if $S(T) > K$ and zero otherwise. This estimator can easily be computed by simulation.

Similarly, differentiating (1) by σ,

$$\frac{\partial S_T}{\partial \sigma} = S_T\left(-\sigma\, T + \sqrt{T}\, Z\right)$$

Solving (1) for Z

$$Z = \frac{1}{\sigma \sqrt{T}} \left(\mathrm{Log}\left(\frac{S_T}{S_0}\right) - \left(r - q - \frac{\sigma^2}{2}\right) T \right)$$

and substituting in the previous equation gives

$$\begin{aligned}
\frac{\partial S_T}{\partial \sigma} &= S_T \left(-\sigma\, T + \frac{\sqrt{T} \left(\mathrm{Log}\left(\frac{S_T}{S_0}\right) - \left(r - q - \frac{\sigma^2}{2}\right) T \right)}{\sigma \sqrt{T}} \right) \\
&= \frac{S_T}{\sigma} \left(-\sigma^2\, T + \mathrm{Log}\left(\frac{S_T}{S_0}\right) - \left(r - q - \frac{\sigma^2}{2}\right) T \right) \\
&= \frac{S_T}{\sigma} \left(\mathrm{Log}\left(\frac{S_T}{S_0}\right) - \left(r - q + \frac{\sigma^2}{2}\right) T \right)
\end{aligned}$$

Combining this with (2) gives the *pathwise estimator of vega* for a European call option

$$\frac{\partial \tilde{V}}{\partial \sigma} = \frac{\partial \tilde{V}}{\partial S_T} \frac{\partial S_T}{\partial \sigma} = e^{-r\,T} \frac{S_T}{\sigma} \boldsymbol{I}\{S(T) > K\} \times \left(\mathrm{Log}\left(\frac{S_T}{S_0}\right) - \left(r - q + \frac{\sigma^2}{2}\right) T \right)$$

Again, this can easily be estimated by simulation.

The pathwise estimator of delta for an arithmetic Asian option is

$$\frac{\partial \tilde{V}}{\partial S_0} = \frac{\partial \tilde{V}}{\partial \overline{S}} \frac{\partial \overline{S}}{\partial S_0}$$

Analogous to the vanilla option (2)

$$\frac{\partial \tilde{V}}{\partial \overline{S}} = e^{-r T} \frac{\mathrm{d}}{\mathrm{d} S_0} \max\left(\overline{S} - K, 0\right) = e^{-r T} \boldsymbol{I}\left\{\overline{S} > K\right\} \tag{3}$$

and

$$\frac{\partial \overline{S}}{\partial S_0} = \frac{1}{m+1} \sum_{i=0}^{m} \frac{d S(t_i)}{d S_0} = \frac{1}{m+1} \sum_{i=0}^{m} \frac{S(t_i)}{S_0} = \frac{\overline{S}}{S_0}$$

Combining the two factors, *the pathwise estimator of delta* for an Asian call option is

$$\frac{\partial \tilde{V}}{\partial S_0} = \frac{\partial \tilde{V}}{\partial \overline{S}} \frac{\partial \overline{S}}{\partial S_0} = e^{-r T} \frac{\overline{S}}{S_0} \boldsymbol{I}\{S(T) > K\}$$

The pathwise method requires continuity in the discounted payoff function as a function of the parameter. For this reason, it is generally not applicable to estimating second derivatives (gamma).

Likelihood ratio

The likelihood ratio method provides an alternative approach, differentiating the probabilities rather than the payoff. Consider a derivative the payoff Y of which depends upon a random variable X whose density $g_\theta(x)$ depends upon some parameter θ. Its expected value is

$$E_\theta[Y] = E_\theta[f(X)\} = \int (f(x)\, g_\theta(x)\, d x$$

Assuming that we can interchange differentiation and integration

$$\frac{d}{d\theta} E_\theta[Y] = \int f(x)\, \dot{g}_\theta(x)\, d x, \qquad \dot{g}_\theta(x) = \frac{d}{d\theta} g_\theta(x)$$

which can be rewritten as

$$\frac{d}{d\theta} E_\theta[Y] = \int f(x)\, \dot{g}_\theta(x)\, d x = \int f(x) \frac{\dot{g}_\theta(x)}{g_\theta(x)} g_\theta(x)\, d x = E_\theta\left[f(X) \frac{\dot{g}_\theta(X)}{g_\theta(X)}\right]$$

Therefore, the expression

$$f(X) \frac{\dot{g}_\theta(X)}{g_\theta(X)}$$

is an unbiased estimator of the derivative of $E_\theta[Y]$. Note that

$$\frac{\dot{g}_\theta(X)}{g_\theta(X)} = \frac{d \log(g_\theta)}{d\theta}$$

is sometimes known as the score function. The score function for delta is

$$\frac{\dot{g}_\theta(X)}{g_\theta(X)} = \frac{Z}{S(0)\, \sigma \sqrt{T}}$$

and therefore the likelihood ratio estimator for delta ($\theta = S_0$) of a European call is

$$\tilde{\Delta} = f(X)\,\frac{\dot{g}_\theta(X)}{g_\theta(X)} = e^{-r\,T}(S(T) - K)^+\,\frac{Z}{S(0)\,\sigma\,\sqrt{T}}$$

Since the score function is the derivative of the density function of S_T, it does not depend on the specific payoff function. Therefore, the likelihood ratio of delta for any option in which the payoff depends only on S_T would take the same form, albeit with a different payoff function.

Similarly the likelihood ratio estimator for vega ($\theta = \sigma$) of a European call is

$$\tilde{\text{vega}} = f(X)\,\frac{\dot{g}_\theta(X)}{g_\theta(X)} = e^{-r\,T}(S(T) - K)^+ = \frac{Z^2 - 1}{\sigma} - \sigma\sqrt{T}$$

The Markov property of GBM implies that S_0 only affects the density of S_1, and has no effect on the density of S_2, S_3 ..., S_m. It follows that the likelihood ratio estimator of delta ($\theta = S_0$) for an Asian call option is

$$\tilde{\tilde{\Delta}} = e^{-r\,T}\left(\overline{S} - K\right)^+\,\frac{Z_1}{S_0\,\sigma\,\sqrt{t_1}}$$

where t_1 is the time to the first observation, and Z_1 is the random variable generating this simulated value.

In contrast to the pathwise method, the likelihood ratio method can be effective in estimating second derivatives, using the estimator

$$f(X)\,\frac{\ddot{g}_\theta(X)}{g_\theta(X)}$$

The likelihood ratio estimator for gamma for a European option is

$$\tilde{\Delta} = f(X)\,\frac{\dot{g}_\theta(X)}{g_\theta(X)} = e^{-r\,T}(S(T) - K)^+\left(\frac{Z^2 - 1}{S_0{}^2\,\sigma^2\,T} - \frac{Z}{S_0{}^2\,\sigma\,\sqrt{T}}\right)$$

Hybrid estimators

We can also combine the pathwise and likelihood ratio methods in a hybrid estimator for gamma. The likelihood ratio estimator for delta for a vanilla call option is

$$\tilde{\Delta} = e^{-r\,T}(S(T) - K)^+\,\frac{Z}{S(0)\,\sigma\,\sqrt{T}}$$

Differentiating with respect to S_0, we obtain the LR-PW estimator for gamma

$$\tilde{\Gamma} = \frac{d}{d\,S_0}\left(e^{-r\,T}(S(T) - K)^+\,\frac{Z}{S_0\,\sigma\,\sqrt{T}}\right) = e^{-r\,T}\,I(S(T) > K)\,\frac{Z}{S_0^2\,\sigma\,\sqrt{T}}\,K$$

Reversing the order, the PW-LR estimator for gamma is

$$\tilde{\Gamma} = e^{-r\,T}\,I(S(T) > K)\,\frac{S_T}{S_0^2}\left(\frac{Z}{\sigma\,\sqrt{T}} - 1\right)$$