The Sobol sequence - an example

The first ten points of a 12-dimensional Sobol sequence are

Out[48]//TableForm=

	1	2	3	4	5	6	7	8	9	10	11	12
1	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
2	1/4	1/4	1/4	3/4	3/4	1/4	3/4	3/4	3/4	3/4	3/4	1/4
3	3/4	3/4	3/4	1/4	1/4	3/4	1/4	1/4	1/4	1/4	1/4	3/4
4	3/8	5/8	7/8	3/8	1/8	3/8	7/8	7/8	5/8	7/8	3/8	3/8
5	7/8	1/8	3/8	7/8	5/8	7/8	3/8	3/8	1/8	3/8	7/8	7/8
6	1/8	7/8	5/8	5/8	7/8	1/8	1/8	1/8	3/8	1/8	5/8	1/8
7	5/8	3/8	1/8	1/8	3/8	5/8	5/8	5/8	7/8	5/8	1/8	5/8
8	5/16	15/16	7/16	9/16	5/16	7/16	15/16	15/16	5/16	11/16	1/16	15/16
9	13/16	7/16	15/16	1/16	13/16	15/16	7/16	7/16	13/16	3/16	9/16	7/16
10	1/16	11/16	3/16	5/16	9/16	3/16	3/16	3/16	9/16	7/16	13/16	11/16

Starting from 250th point, the next 10 points are

Out[56]//TableForm=

	1	2	3	4	5	6	7	8	9	10	11	12
250	31/256	229/256	243/256	125/256	91/256	3/256	77/256	17/256	123/256	67/256	203/256	211/256
251	159/256	101/256	115/256	253/256	219/256	131/256	205/256	145/256	251/256	195/256	75/256	83/256
252	63/256	5/256	83/256	221/256	187/256	35/256	109/256	49/256	27/256	99/256	107/256	243/256
253	191/256	133/256	211/256	93/256	59/256	163/256	237/256	177/256	155/256	227/256	235/256	115/256
254	127/256	69/256	19/256	29/256	123/256	99/256	173/256	241/256	219/256	163/256	171/256	179/256
255	255/256	197/256	147/256	157/256	251/256	227/256	45/256	113/256	91/256	35/256	43/256	51/256
256	255/512	347/512	151/512	399/512	55/512	187/512	183/512	147/512	209/512	233/512	511/512	121/512
257	511/512	91/512	407/512	143/512	311/512	443/512	439/512	403/512	465/512	489/512	255/512	377/512
258	127/512	475/512	23/512	15/512	439/512	59/512	311/512	275/512	337/512	361/512	127/512	249/512
259	383/512	219/512	279/512	271/512	183/512	315/512	55/512	19/512	81/512	105/512	383/512	505/512

This 12-dimensional sequence was generated using the following primitive polynomials and initial values (Joe and Kuo 2007)

Out[74]=

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{1 + x, {1}}
{1 + x + x^2, {1, 3}}
{1 + x + x^3, {1, 3, 1}}
{1 + x^2 + x^3, {1, 1, 1}}
{1 + x + x^4, {1, 1, 3, 3}}
{1 + x^3 + x^4, {1, 3, 5, 13}}
{1 + x^2 + x^5, {1, 1, 5, 5, 17}}
{1 + x^3 + x^5, {1, 1, 5, 5, 5}}
{1 + x + x^2 + x^3 + x^5, {1, 1, 7, 11, 19}}
{1 + x + x^2 + x^4 + x^5, {1, 1, 5, 1, 1}}
{1 + x + x^3 + x^4 + x^5, {1, 1, 1, 3, 11}}
{1 + x^2 + x^3 + x^4 + x^5, {1, 3, 5, 5, 31}}
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Each primitive polynomial defines a unique recursive function, which is used to generate a set of direction numbers for that dimension. For example, the 6th dimension is based upon the 4th degree polynomial $x^4 + x^3 + 1$ which defines the 4th order recursion

$$m_j = 2\,a_1\,m_{j-1} \oplus 2^2\,a_2\,m_{j-2} \oplus 2^3\,a_3\,m_{j-3} \oplus m_{j-4}$$

which collapses to

$$m_j = 2\,m_{j-1} \oplus m_{j-4}$$

since $a_1 = 1$ while $a_2 = a_3 = 0$ (a_i is the coefficient of x^{k-i}, where k is the order of the polynomial).

Seeded with k non-negative odd integers $m_1, m_2, m_3, \ldots$ with $m_i < 2^i$, subsequent values generated by the recursion are also non-negative odd integers with $m_j < 2^j$ for all j. Direction numbers are obtained by dividing each term by 2^j, that is

$$(v_1, v_2, \ldots, v_M) = \left(\frac{m_1}{2}, \frac{m_2}{4}, \frac{m_3}{8}, \ldots, \frac{m_M}{2^M}\right)$$

To produce a Sobol sequence of length n requires one direction number for each bit in the binary expansion of n, a total of $k = \lceil \mathrm{Log}_2\, n \rceil$ direction numbers for each dimension.

Using the above initial values, the direction numbers of a 12-dimensional Sobol sequence starts with

Out[80]//TableForm=

	1	2	3	4	5	6	7	8	9	10	11	12
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{1}{4096}$
2	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{3}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$	$\frac{1}{2048}$	$\frac{3}{4096}$
3	$\frac{5}{2}$	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{32}$	$\frac{5}{64}$	$\frac{5}{128}$	$\frac{5}{256}$	$\frac{7}{512}$	$\frac{5}{1024}$	$\frac{1}{2048}$	$\frac{5}{4096}$
4	$\frac{15}{2}$	$\frac{9}{4}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{32}$	$\frac{13}{64}$	$\frac{5}{128}$	$\frac{5}{256}$	$\frac{11}{512}$	$\frac{1}{1024}$	$\frac{3}{2048}$	$\frac{5}{4096}$
5	$\frac{17}{2}$	$\frac{29}{4}$	$\frac{31}{8}$	$\frac{31}{16}$	$\frac{25}{32}$	$\frac{11}{64}$	$\frac{17}{128}$	$\frac{5}{256}$	$\frac{19}{512}$	$\frac{1}{1024}$	$\frac{11}{2048}$	$\frac{31}{4096}$
6	$\frac{51}{2}$	$\frac{23}{4}$	$\frac{29}{8}$	$\frac{55}{16}$	$\frac{9}{32}$	$\frac{37}{64}$	$\frac{9}{128}$	$\frac{53}{256}$	$\frac{37}{512}$	$\frac{27}{1024}$	$\frac{43}{2048}$	$\frac{35}{4096}$
7	$\frac{85}{2}$	$\frac{71}{4}$	$\frac{81}{8}$	$\frac{61}{16}$	$\frac{43}{32}$	$\frac{31}{64}$	$\frac{9}{128}$	$\frac{53}{256}$	$\frac{69}{512}$	$\frac{79}{1024}$	$\frac{75}{2048}$	$\frac{113}{4096}$
8	$\frac{255}{2}$	$\frac{197}{4}$	$\frac{147}{8}$	$\frac{157}{16}$	$\frac{251}{32}$	$\frac{227}{64}$	$\frac{45}{128}$	$\frac{113}{256}$	$\frac{91}{512}$	$\frac{35}{1024}$	$\frac{43}{2048}$	$\frac{51}{4096}$
9	$\frac{257}{2}$	$\frac{209}{4}$	$\frac{433}{8}$	$\frac{181}{16}$	$\frac{449}{32}$	$\frac{381}{64}$	$\frac{237}{128}$	$\frac{113}{256}$	$\frac{103}{512}$	$\frac{175}{1024}$	$\frac{425}{2048}$	$\frac{31}{4096}$
10	$\frac{771}{2}$	$\frac{627}{4}$	$\frac{149}{8}$	$\frac{191}{16}$	$\frac{449}{32}$	$\frac{143}{64}$	$\frac{633}{128}$	$\frac{353}{256}$	$\frac{871}{512}$	$\frac{695}{1024}$	$\frac{37}{2048}$	$\frac{133}{4096}$

This set is sufficient to produce 1024 random points from the 12-dimensional hypercube. Successive vectors in the Sobol sequence are produced by XORing the preceding vector with one of the rows in the above table, where the rows are chosen in odometer order starting with the first row.