Spread options

- **Preliminaries**

Definition

A spread option is an option whose payoff depends upon the *difference* in the prices of two or more assets.

$$\text{payoff} = (S_1 - S_2 - K)^+$$

Examples

- **Commodity markets**

The soybean *crush spread* traded on the CBOT is

$$\text{CS} = \frac{48}{2000} S^M + \frac{11}{100} S^O - S^B$$

where S^M is the price is the futures price of soy meal in dollars per ton, S^O is the futures price of soy oil in dollars per 100 pounds, and S_t is the futures price of soybean in dollars per bushel. The payoff of a call option on the soybean crush is

$$\text{CSC} = (\text{CS} - K)^+$$

- **Energy markets**

The payoff of the 1:1:0 gasoline *crack spread* call is

$$\text{CSGC} = (42\ \text{UG} - \text{CO} - K)^+$$

where UG is the price of unleaded gasoline (\$ / gallon) and CO is the price of crude oil (\$ / barrel). The payoff of the 3:2:1 *crack spread* call is

$$\text{CSHC} = \left(42\left(\frac{2}{3}\ \text{UG} + \frac{1}{3}\ \text{HO}\right) - \text{CO} - K\right)^+$$

where HO is the price of heating oil. Crack spread options are traded on NYMEX.

A *spark spread* is a proxy for cost of converting a specific fuel (usually natural gas) into electricity.

$$\text{SS} = E - H_{\text{eff}}\ \text{NG}$$

where E and NG are the futures prices of electricity and natural gas, and H_{eff} is the heat rate.

- **Interest rate markets**

The *TED* spread measures the difference between 3-month Tbills and 3-month LIBOR. The *MOB* spread measures the difference in yield between municipal and treasury bonds.

Exchange options

A spread option with a zero strike is known as an option to exchange

$$\text{payoff} = (S_1 - S_2)^+$$

for which we have an exact formula (known as the Magrabe formula).

$$c = e^{-q_1 T} S_1 N(d_1) - e^{-q_2 T} S_2 N(d_2)$$

where

$$d_1 = \frac{\mathrm{Ln}(S_1/S_2) + \left(q_2 - q_1 + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\sigma = \sqrt{\sigma_1^2 - 2\,\rho\,\sigma_1\,\sigma_1 + \sigma_2^2}$$

Note that the value is independent of the risk-free rate. Alternatively, in terms of futures prices, we have

$$c = e^{-r\,T}\left(F_1\,N(d_1) - F_2\,N(d_2)\right)$$

where

$$d_1 = \frac{\mathrm{Ln}(F_1/F_2) + \sigma^2\,T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\sigma = \sqrt{\sigma_1^2 - 2\,\rho\,\sigma_1\,\sigma_1 + \sigma_2^2}$$

Approximations

- **The Kirk approximation**

The payoff of a spread option is

$$\text{payoff} = (S_1 - S_2 - K)^+$$

The popular Kirk approximation values the spread option as an exchange between F_1 and $F_2 + K$, treating $F_2 + K$ as lognormal with volatilty $\left(\frac{F_2}{F_2+K}\right)\sigma_2$.

$$c = e^{-r\,T}\left(F_1\,N(d_1) - (F_2 + K)\,N(d_2)\right)$$

where

$$F_i = e^{(r-q_i)\,T}\,S_i,\; i = 1, 2$$

$$\sigma = \sqrt{\sigma_1^2 - 2\,b\,\rho\,\sigma_1\,\sigma_1 + b^2\,\sigma_2^2}\,, \quad b = \frac{F_2}{F_2 + K}$$

$$d_1 = \frac{\mathrm{Ln}\left(\frac{F_1}{F_2+K}\right) + \frac{1}{2}\sigma^2\,T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Clearly, the Kirk approximation reduces to the Margrabe formula when $K = 0$, and will be most useful when $K << S_2$.

- **Bjerksund and Stensland**

The payoff of the spread can be written as

$$C(T) = (S_1 - S_2 - K)^+ = (S_1 - S_2 - K)\cdot I\left(S_1(T) \geq S_2(T) + K\right)$$

where $I(\,)$ is the indicator function, taking the value one when its argument is true, and zero otherwise. Bjerksund and Stensland consider the related derivative with the payoff

$$c(T) = (S_1 - S_2 - K) \cdot I\left(S_1(T) \geq \frac{a\,(S_2(T))^b}{E\left[(S_2(T))^b\right]}\right)$$

where $a = F_2 + K$, $b = F_2/(F_2 + K)$ and F_2 is the forward price $F_2 = S_2\, e^{(r-q_2)\,T}$. This has two implications:

- They can compute the exact value of the related derivative, which they propose as a superior alternative to the Kirk approximation.
- The related derivative can be used as a control variable in simulating the value of a spread option.

The exact value is given by the following formula.

$$c = e^{-r\,T}\left(F_1\, N(d_1) - F_2\, N(d_2) - K\, N(d_3)\right)$$

where

$$F_i = e^{(r-q_i)\,T}\, S_i,\ i = 1, 2$$

$$\sigma = \sqrt{\sigma_1^2 - 2\, b\, \rho\, \sigma_1\, \sigma_1 + b^2\, \sigma_2^2}$$

$$d_1 = \frac{\mathrm{Ln}\left(\frac{F_1}{a}\right) + \left(\frac{1}{2}\sigma_1^2 - b\,\rho\,\sigma_1\,\sigma_2 + \frac{1}{2} b^2\,\sigma_2^2\right) T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\mathrm{Ln}\left(\frac{F_1}{a}\right) + \left(-\frac{1}{2}\sigma_1^2 + \rho\,\sigma_1\,\sigma_2 + \frac{1}{2} b^2\,\sigma_2^2 - b\,\sigma_2^2\right) T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \frac{\left(\sigma 1^2 - (1+b)\,\rho\,\sigma 1\,\sigma 2 + b\,\sigma 2^2\right)}{\sigma}\sqrt{T}$$

$$d_3 = d_1 - \frac{\left(\sigma 1^2 - b\,\rho\,\sigma 1\,\sigma 2\right)}{\sigma}\sqrt{T}$$

$$d_3 = \frac{\mathrm{Ln}\left(\frac{F_1}{a}\right) + \left(-\frac{1}{2}\sigma_1^2 + \frac{1}{2} b^2\,\sigma_2^2\right) T}{\sigma\sqrt{T}}$$

Greeks

For the BjS

$$c = e^{-r\,T}\left(F_1\, N(d_1) - F_2\, N(d_2) - K\, N(d_3)\right)$$

First note

$$\frac{\partial d_i}{\partial F_1} = \frac{1}{F_1\,\sigma\sqrt{T}}$$

$$\frac{\partial c}{\partial F_1} = e^{-r\,T}\left(N(d_1) + (F_1\,\phi(d_1) - F_2\,\phi(d_2) - K\,\phi(d_3))\,\frac{\partial d_i}{\partial F_1}\right)$$

$$= e^{-r\,T}\left(N(d_1) + \frac{F_1\,\phi\,(d_1) - F_2\,\phi_1(d_2) - K\,\phi\,(d_3)}{F_1\,\sigma\sqrt{T}}\right)$$

and therefore

$$\frac{\partial c}{\partial S_1} = \frac{\partial c}{\partial F_1}\,\frac{\partial F_1}{\partial S_1} = e^{-q\,T}\left(N(d_1) + \frac{F_1\,\phi\,(d_1) - F_2\,\phi_1(d_2) - K\,\phi\,(d_3)}{F_1\,\sigma\sqrt{T}}\right)$$

Computing the derivative with respect to F_2 is a a little more difficult, since the adjusted volatility depends upon b, which is a function of F_2.

$$\frac{\partial b}{\partial F_2} = \frac{b^2 K}{F_2^2}$$

$$\frac{\partial d_0}{\partial F_2} = \frac{1}{2\sqrt{T}\ \sigma^3}\left(\sigma 2\,(\rho\,\sigma 1 - b\,\sigma 2)\,\frac{K\,b^2}{\mathrm{F2}^2}\left(\mathrm{Log}\left[\frac{\mathrm{F1}}{\mathrm{F2}+K}\right] - T\,\sigma^2\right) - \frac{2\,\sigma^2}{\mathrm{F2}+K}\right)$$

$$\frac{\partial d_1}{\partial F_2} = \frac{1}{2\sqrt{T}\ \sigma^3}\left(\sigma 2\,(\rho\,\sigma 1 - b\,\sigma 2)\,\frac{K\,b^2}{\mathrm{F2}^2}\left(2\,\mathrm{Log}\left[\frac{\mathrm{F1}}{\mathrm{F2}+K}\right] - T\,\sigma^2\right) - \frac{2\,\sigma^2}{\mathrm{F2}+K}\right)$$

$$\frac{\partial d_2}{\partial F_2} = \frac{\partial d_1}{\partial F_2} - \frac{b^2\,K}{\mathrm{F2}^2}\,\frac{(1-b)\,\left(1-\rho^2\right)\sigma 1^2\,\sigma 2^2}{\sigma^3}\,\sqrt{T}$$