

Spread options

- Preliminaries

Definition

A spread option is an option whose payoff depends upon the *difference* in the prices of two or more assets.

$$\text{payoff} = (S_1 - S_2 - K)^+$$

Examples

- Commodity markets

The soybean *crush spread* traded on the CBOT is

$$CS = \frac{48}{2000} S^M + \frac{11}{100} S^O - S^B$$

where S^M is the price is the futures price of soy meal in dollars per ton, S^O is the futures price of soy oil in dollars per 100 pounds, and S^B is the futures price of soybean in dollars per bushel. The payoff of a call option on the soybean crush is

$$CSC = (CS - K)^+$$

- Energy markets

The payoff of the 1:1:0 gasoline *crack spread* call is

$$CSGC = (42 \text{ UG} - \text{CO} - K)^+$$

where UG is the price of unleaded gasoline (\$ / gallon) and CO is the price of crude oil (\$ / barrel). The payoff of the 3:2:1 *crack spread* call is

$$CSHC = \left(42 \left(\frac{2}{3} \text{UG} + \frac{1}{3} \text{HO} \right) - \text{CO} - K \right)^+$$

where HO is the price of heating oil. Crack spread options are traded on NYMEX.

A *spark spread* is a proxy for cost of converting a specific fuel (usually natural gas) into electricity.

$$SS = E - H_{\text{eff}} \text{ NG}$$

where E and NG are the futures prices of electricity and natural gas, and H_{eff} is the heat rate.

- Interest rate markets

The *TED* spread measures the difference between 3-month Tbills and 3-month LIBOR. The *MOB* spread measures the difference in yield between municipal and treasury bonds.

Exchange options

A spread option with a zero strike is known as an option to exchange

$$\text{payoff} = (S_1 - S_2)^+$$

for which we have an exact formula (known as the Magrabe formula).

$$c = e^{-q_1 T} S_1 N(d_1) - e^{-q_2 T} S_2 N(d_2)$$

where

$$d_1 = \frac{\text{Ln}(S_1/S_2) + (q_2 - q_1 + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

Note that the value is independent of the risk-free rate. Alternatively, in terms of futures prices, we have

$$c = e^{-rT} (F_1 N(d_1) - F_2 N(d_2))$$

where

$$d_1 = \frac{\text{Ln}(F_1/F_2) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

Approximations

■ The Kirk approximation

The payoff of a spread option is

$$\text{payoff} = (S_1 - S_2 - K)^+$$

The popular Kirk approximation values the spread option as an exchange between F_1 and $F_2 + K$, treating $F_2 + K$ as lognormal with volatility $\left(\frac{F_2}{F_2+K}\right)\sigma_2$.

$$c = e^{-rT} (F_1 N(d_1) - (F_2 + K) N(d_2))$$

where

$$F_i = e^{(r-q_i)T} S_i, \quad i = 1, 2$$

$$\sigma = \sqrt{\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2}, \quad b = \frac{F_2}{F_2 + K}$$

$$d_1 = \frac{\text{Ln}\left(\frac{F_1}{F_2+K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Clearly, the Kirk approximation reduces to the Margrabe formula when $K = 0$, and will be most useful when $K \ll S_2$.

■ Bjerk Sund and Stensland

The payoff of the spread can be written as

$$C(T) = (S_1 - S_2 - K)^+ = (S_1 - S_2 - K) \cdot I(S_1(T) \geq S_2(T) + K)$$

where $I(\cdot)$ is the indicator function, taking the value one when its argument is true, and zero otherwise. Bjerk Sund and Stensland consider the related derivative with the payoff

$$c(T) = (S_1 - S_2 - K) \cdot I \left(S_1(T) \geq \frac{a (S_2(T))^b}{E[(S_2(T))^b]} \right)$$

where $a = F_2 + K$, $b = F_2/(F_2 + K)$ and F_2 is the forward price $F_2 = S_2 e^{(r-q_2)T}$. This has two implications:

- They can compute the exact value of the related derivative, which they propose as a superior alternative to the Kirk approximation.
- The related derivative can be used as a control variable in simulating the value of a spread option.

The exact value is given by the following formula.

$$c = e^{-rT} (F_1 N(d_1) - F_2 N(d_2) - K N(d_3))$$

where

$$F_i = e^{(r-q_i)T} S_i, \quad i = 1, 2$$

$$\sigma = \sqrt{\sigma_1^2 - 2b\rho\sigma_1\sigma_2 + b^2\sigma_2^2}$$

$$d_1 = \frac{\text{Ln}\left(\frac{F_1}{a}\right) + \left(\frac{1}{2}\sigma_1^2 - b\rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\text{Ln}\left(\frac{F_1}{a}\right) + \left(-\frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}b^2\sigma_2^2 - b\sigma_2^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \frac{(\sigma_1^2 - (1+b)\rho\sigma_1\sigma_2 + b\sigma_2^2)}{\sigma} \sqrt{T}$$

$$d_3 = d_1 - \frac{(\sigma_1^2 - b\rho\sigma_1\sigma_2)}{\sigma} \sqrt{T}$$

$$d_3 = \frac{\text{Ln}\left(\frac{F_1}{a}\right) + \left(-\frac{1}{2}\sigma_1^2 + \frac{1}{2}b^2\sigma_2^2\right)T}{\sigma\sqrt{T}}$$

Greeks

For the BJS

$$c = e^{-rT} (F_1 N(d_1) - F_2 N(d_2) - K N(d_3))$$

First note

$$\frac{\partial d_i}{\partial F_1} = \frac{1}{F_1 \sigma \sqrt{T}}$$

$$\begin{aligned} \frac{\partial c}{\partial F_1} &= e^{-rT} \left(N(d_1) + (F_1 \phi(d_1) - F_2 \phi(d_2) - K \phi(d_3)) \frac{\partial d_i}{\partial F_1} \right) \\ &= e^{-rT} \left(N(d_1) + \frac{F_1 \phi(d_1) - F_2 \phi(d_2) - K \phi(d_3)}{F_1 \sigma \sqrt{T}} \right) \end{aligned}$$

and therefore

$$\frac{\partial c}{\partial S_1} = \frac{\partial c}{\partial F_1} \frac{\partial F_1}{\partial S_1} = e^{-qT} \left(N(d_1) + \frac{F_1 \phi(d_1) - F_2 \phi(d_2) - K \phi(d_3)}{F_1 \sigma \sqrt{T}} \right)$$

Computing the derivative with respect to F_2 is a little more difficult, since the adjusted volatility depends upon b , which is a function of F_2 .

$$\frac{\partial b}{\partial F_2} = \frac{b^2 K}{F_2^2}$$

$$\frac{\partial d_0}{\partial F_2} = \frac{1}{2\sqrt{T} \sigma^3} \left(\sigma^2 (\rho \sigma_1 - b \sigma_2) \frac{K b^2}{F_2^2} \left(\text{Log} \left[\frac{F_1}{F_2 + K} \right] - T \sigma^2 \right) - \frac{2 \sigma^2}{F_2 + K} \right)$$

$$\frac{\partial d_1}{\partial F_2} = \frac{1}{2\sqrt{T} \sigma^3} \left(\sigma^2 (\rho \sigma_1 - b \sigma_2) \frac{K b^2}{F_2^2} \left(2 \text{Log} \left[\frac{F_1}{F_2 + K} \right] - T \sigma^2 \right) - \frac{2 \sigma^2}{F_2 + K} \right)$$

$$\frac{\partial d_2}{\partial F_2} = \frac{\partial d_1}{\partial F_2} - \frac{b^2 K}{F_2^2} \frac{(1-b)(1-\rho^2)\sigma_1^2 \sigma_2^2}{\sigma^3} \sqrt{T}$$