Asian options

Introduction

Asian options are popular in currency and commodity markets because

- they offer a cheaper method of hedging exposure to regular periodic cash flows
- they are less susceptible to manipulation of the spot market.

There are two classes of Asian options

- average price options (payoff $\max\left(\overline{S} - K, 0\right)$ or $\max\left(K - \overline{S}, 0\right)$)
- average strike options (payoff $\max\left(S_T - \overline{S}, 0\right)$ or $\max\left(\overline{S} - S_T, 0\right)$)

Average strike options are sometimes called floating strike options. Vanmaele et al. (2006) provide transformations between average price and average strike options.

In addition, there are two ways of calculating the average $\overline{S}$ - arithmetic and geometric.

$$A = \frac{1}{m+1}(S_0 + S_2 + \ldots + S_m) \quad \geq \quad G = (S_0 \times S_1 \times \ldots \times S_n)^{\frac{1}{m+1}}$$

Note that by convention, the average includes the price on the first day of averaging. Jensen's inequality states that the arithmetic mean is larger than the geometric mean, with equality if and only if the observations are equal.

Unfortunately, most Asian options in practice are based on arithmetic averaging, while precise results are readily available only for geometric averages. In practice, we use the price of an equivalent geometric average option as a lever to price its arithmetic counterpart.

Upper and lower bounds

Jensen's inequality implies that the value of an arithmetic Asian option is bounded below by its geometric counterpart.

$$c_G(T) \leq c_A(T)$$

We can also develop useful upper bounds. The value of the arithmetic Asian option is

$$c_A(T) = e^{-rT} E\left[\left(\overline{S}_T - K\right)_+\right]$$

where expectation is taken with respect to the risk neutral distribution. This implies

$$\begin{aligned}
c_A(T) &= e^{-rT} E\left[\left(\overline{S}_T - K\right)_+\right] \\
&= e^{-rT} E\left[\left(\frac{1}{m+1}\sum_{i=0}^{m} S(t_i) - K\right)_+\right] \\
&= \frac{1}{m+1} e^{-rT} E\left[\left(\sum_{i=0}^{m}(S(t_i) - K)\right)_+\right] \\
&\leq \frac{1}{m+1} e^{-rT} E\left[\sum_{i=0}^{m}(S(t_i) - K)_+\right] \\
&= \frac{1}{m+1} \sum_{i=0}^{m} e^{-rT} E[(S(t_i) - K)_+]
\end{aligned}$$

$$\leq \frac{1}{m+1}\sum_{i=0}^{m} c(t_i)$$

The value of an Asian option is less than the value of a portfolio of intermediate options with the same strike price. Furthermore, absent dividends, $c(t_i) \leq c(T)$ and therefore

$$c_G(T) \leq c_A(T) \leq \frac{1}{m+1}\sum_{i=0}^{m} c(t_i) \leq c(T)$$

- **Put-call parity**

Put-call parity applies to European average price options, whether arithmetic or geometric. To see this, observe that the difference between the terminal values of a put and call option is

$$c(0) - p(0) = \left(\overline{S} - K\right)_+ - \left(K - \overline{S}\right)_+ = \overline{S} - K$$

Taking risk-neutral expectations

$$c(T) - p(T) = e^{-r\,T}\left(E[\overline{S}] - K\right)$$

The expectation $E[\overline{S}]$ can be calculated exactly for both arithmetic and geometric averages. Consequently, it suffices to compute call values and derive the values of puts from (1).

Geometric average options

Suppose that S_t is lognormal with mean $\nu\,t$ and variance $\sigma^2\,t$. Then the geometric average

$$G_m = (S_0 \times S_1 \times \ldots \times S_m)^{\frac{1}{m+1}}$$

is lognormal with mean

$$E[G_m] = S_0\, e^{\left(\nu + \frac{2m+1}{m+1}\frac{\sigma^2}{6}\right)\frac{T}{2}} \tag{1}$$

and variance

$$V[\log G_m] = \left(\frac{2\,m+1}{m+1}\right)\frac{\sigma^2}{6}\,T \equiv s^2 \tag{2}$$

See Appendix. Under the risk neutral distribution $\nu = r - q - \sigma^2/2$. Substituting in (1)

$$E[G_m] = S_0\, e^{\left(r - q - \frac{m+2}{m+1}\frac{\sigma^2}{6}\right)\frac{T}{2}}$$

By the fundamental theorem (Hull 2003: 262), the value of a European call option on G_m is

$$c_G = e^{-r\,t}\,E[\max(G_m - K, 0)] = e^{-r\,t}(E(G_m)\,N(d_1) - K\,N(d_2))$$

where $N()$ denotes the standard normal distribution function and

$$d_1 = \frac{\ln\left(\frac{E[G_m]}{K}\right) + \frac{s^2}{2}}{s},\quad d_2 = \frac{\ln\left(\frac{E[G_m]}{K} - \frac{s^2}{2}\right)}{s}$$

With continuous averaging, $m \longrightarrow \infty$ and

$$E[G_\infty] = S_0\, e^{\left(r - q - \frac{\sigma^2}{6}\right)\frac{T}{2}}$$

with

$$V[\log G_\infty] = \frac{\sigma^2}{3}\,T$$

Making the substitution

$$q^* = \frac{1}{2}\left(r + q + \frac{\sigma^2}{6}\right)$$

the expected value is

$$E[G_\infty] = S_0\, e^{\left(r - q - \frac{\sigma^2}{6}\right)\frac{T}{2}} = S_0\, e^{(r - q^*)\,T}$$

which is the same as the expected value of S_T assuming the dividend yield q^*.

Consequently, the value of European geometric Asian option with continuous averaging is given by Black-Scholes formula with the substitutions (Hull 2003, p. 444)

$$\text{dividend yield} = \frac{1}{2}\left(r + q + \frac{\sigma^2}{6}\right) \qquad \text{volatility} = \frac{\sigma}{\sqrt{3}}$$

This also provides a useful approximation for large n.

Moments of the geometric mean under the risk neutral distribution

□	Discrete	Continuous
$E[G]$	$S_0\, e^{\left(r - q - \frac{m+2}{m+1}\frac{\sigma^2}{6}\right)\frac{T}{2}}$	$S_0\, e^{\left(r - q - \frac{\sigma^2}{6}\right)\frac{T}{2}}$
$V[\log G]$	$\left(\frac{2m+1}{m+1}\right)\frac{\sigma^2}{6}\,T$	$\frac{\sigma^2}{3}\,T$

Greeks for the geometric average option

Delta for the geometric average option is

$$\Delta = \delta\left(\frac{E[\text{Gm}]}{S_0}\right) N(d_1)$$

where

$$d_1 = \frac{\log\left(\frac{E[\text{Gm}]}{K}\right) + \frac{s^2}{2}}{s}$$

To derive gamma, we have

$$E[G_m] = S_0\, e^{\left(\nu + \frac{2m+1}{m+1}\frac{\sigma^2}{6}\right)\frac{T}{2}}$$

so that

$$\frac{d\,E[G_m]}{d\,S_0} = e^{\left(\nu + \frac{2m+1}{m+1}\frac{\sigma^2}{6}\right)\frac{T}{2}} = \frac{E[G_m]}{S_0}$$

and

$$\frac{d}{d\,S_0}\left(\frac{E[G_m]}{S_0}\right) = \frac{S_0\left(\frac{E[G_m]}{S_0}\right) - E[G_m]}{S_0^2} = 0$$

That is, the expected return is independent of the starting point.

$$\frac{d}{d\,S_0}(d_1) = \frac{1}{s\,S_0}$$

By the chain rule

$$\frac{d}{d\,S_0}(N(d_1)) = N'(d_1)\,\frac{1}{s\,S_0}$$

where N' is the normal density function. Combining these derivations

$$\Gamma = \frac{d\Delta}{dS_0} = \delta\left(\frac{E[\mathrm{Gm}]}{S_0}\right)\frac{d}{dS_0}(N(d_1)) + N(d_1)\,\frac{d}{dS_0}\left(\frac{E[G_m]}{S_0}\right) = \delta\left(\frac{E[\mathrm{Gm}]}{S_0}\right)N'(d_1)\,\frac{1}{s\,S_0}$$

□	Discrete	Continuous
Δ	$\delta\left(\frac{E[\mathrm{Gm}]}{S_0}\right) N(d_1)$	□
Γ	$\delta\left(\frac{E[\mathrm{Gm}]}{S_0}\right) N'(d_1)\,\frac{1}{s\,S_0}$	□

Arithmetic average options

The arithmetic average of lognormal random variables is not lognormal, and its precise distribution has proved intractable. There are three practical approaches to accurate valuation of arithmetic Asian options:

- analytical approximation
- simulation using the geometric average option as a control variate
- a modified binomial method

Analytical approximation

Although the distribution of the arithmetic average is A is intractable, its moments $E\left[A^k\right]$ can be readily calculated. This has spurred a variety of approximation methods.

Moments of A

Observe that

$S_1 = S_0\,R_1$

$S_2 = S_1\,R_2 = S_0\,R_1\,R_2$

$S_m = S_0\,R_1\,R_2\,\ldots\,R_n$

$$\sum_{i=0}^{m} S_i = S_0(1 + R_1 + R_1\,R_2 + \ldots + R_1\,R_2\ldots R_m)$$

where the gross returns R_i are independent and identically distributed. Let $a = E\,[R_i]$. Then

$$E\left[\sum_{i=0}^{m} S_i\right] = S_0\,E[1 + R_1 + R_1\,R_2 + \ldots + R_1\,R_2\ldots R_n] = S_0\left(1 + a + a^2 + \ldots + a^m\right) = S_0\,\frac{1 - a^{m+1}}{1 - a}$$

Under the risk neutral distribution, $a = E[R_i] = e^{(r-q)\,T/m}$. Therefore

$$E[A_m] = \frac{1}{m+1}\,E\left[\sum_{i=0}^{m} S_i\right] = \frac{S_0}{m+1}\,\frac{1 - e^{(r-q)\,T(m+1)/m}}{1 - e^{(r-q)\,T/m}}$$

For continuous averaging, we have (Hull 2003: 444)

$$E[A_\infty] = \lim_{n\to\infty} E[A_n] = \lim_{m\to\infty} \frac{S_0}{m+1}\,\frac{1 - e^{(r-q)\,T(m+1)/m}}{1 - e^{(r-q)\,T/m}} = \frac{\left(e^{(r-q)\,T} - 1\right)}{(r-q)\,T}\,S_0$$

An upper bound

We have previous observed that $G \leq A$ (Jensen's inequality). This implies that $C_G \leq G_A$, but also that the difference in payoffs is

$$(A-K)^+ - (G-K)^+ = \begin{pmatrix} 0, & G \leq A \leq K \\ A-K, & G \leq K \leq A \\ A-G, & K \leq G \leq A \end{pmatrix}$$

In every case we have

$$(A-K)^+ - (G-K)^+ \leq A - G$$

Taking risk-neutral expectations

$$C_A(T) - C_G(T) \leq e^{-rT}(E[A] - E[G])$$

and therefore

$$C_G(T) \leq C_A(T) \leq C_G(T) + e^{-rT}(E[A] - E[G])$$

Example. For $S_0 = 50$, $r = 10\,\%$, $q = 0$, $\sigma = 40\,\%$, $T = 1$ and $n = 250$, the expected values of the arithmetic and geometric means are 52.59 and 51.86 respectively. The value of geometric average call option is 5.13. Therefore, we can deduce that the value of an arithmetic average option C_A is bounded as follows

$$5.13 \leq C_A \leq 5.13 + e^{-0.1}(52.37 - 51.87) = 5.79$$

- **Simple modified geometric**

The simplest analytical approximation assumes that arithmetic average is lognormally distributed with mean $E[A_n]$, but assuming the standard deviation of $\ln A_n$ is $\sigma_g = \sigma/\sqrt{3}$. By the fundamental theorem (Hull 2003: 262)

$$C_A = e^{-rT} E[\max(S-K, 0)] = e^{-rT}(E[A_n]\, N(d_1) - K\, N(d_2))$$

where

$$d_1 = \frac{\ln(E[A_n]/K) + \sigma_G^2\, T/2}{\sigma_G \sqrt{T}}, \; d_2 = d_1 - \sigma_G \sqrt{T}$$

- **Other analytical approximations**

Extensions of the previous idea include:

- **Levy**: Assume A_n is lognormally distributed with true mean *and* variance (Hull 2003: 444)
- **Turnbull and Wakeman**: Approximate the actual distribution of A_n using Edgeworth expansions.

There is evidence that Levy approximation is not generally more accurate (and may be less accurate) than the simple modified geometic method, and that the Turnbull and Wakeman method is no more accurate unless higher (third and fourth) moments are used (James 2003: 215-216).

- **Bounds**

Earlier, we developed upper and lower bounds for an arithmetic average option

$$C_G(T) \leq C_A(T) \leq C_G(T) + e^{-rT}(E[A] - E[G])$$

In a recent contribution, Nielsen and Sandmann (2003) develop tighter upper and lower bounds that are easily calculated. In a numerical analysis of 32 options, they find that the Levy approximation satisfies the bounds in only eight cases while the Turnbull and Wakeman method satisfies the bounds in only seven cases. They conclude "more information is gained from the easily calculated bounds than from the pricing approximations" in their sample. They also show how the bounds can be used to calculate hedge parameters.