Chapter 6: Comparative Statics

6.1 The Jacobian is

$$J = \begin{pmatrix} H_L & J_{\mathbf{g}}^T \\ J_{\mathbf{g}} & 0 \end{pmatrix}$$

where H_L is the Hessian of the Lagrangean. We note that

- $H_L(\mathbf{x}_0)$ is negative definite in the subspace $T = \{\, \mathbf{x} : J_{\mathbf{g}}\mathbf{x} = \mathbf{0} \,\}$ (since $\mathbf{x}_0$ satisfies the conditions for a strict local maximum)
- $J_{\mathbf{g}}$ has rank m (since the constraints are regular).

Consider the system of equations

$$\begin{pmatrix} H_L & J_{\mathbf{g}}^T \\ J_{\mathbf{g}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{6.1}$$

where $\mathbf{x} \in \Re^n$ and $\mathbf{y} \in \Re^m$. It can be decomposed into

$$H_L\mathbf{x} + J_{\mathbf{g}}^T\mathbf{y} = \mathbf{0} \tag{6.2}$$

$$J_{\mathbf{g}}\mathbf{x} = \mathbf{0} \tag{6.3}$$

Suppose $\mathbf{x}$ solves (6.3). Multiplying (6.2) by $\mathbf{x}^T$ gives

$$\mathbf{x}^T H_L \mathbf{x} + \mathbf{x}^T J_{\mathbf{g}}^T \mathbf{y} = \mathbf{x}^T H_L \mathbf{x} + (J_{\mathbf{g}}\mathbf{x})^T \mathbf{y} = \mathbf{0}$$

But (6.3) implies that the second term is $\mathbf{0}$ and therefore $\mathbf{x}^T H_L \mathbf{x} = \mathbf{0}$. Since H_L is positive definite on $T = \{\, \mathbf{x} : J_{\mathbf{g}}\mathbf{x} = \mathbf{0} \,\}$, we must have $\mathbf{x} = 0$. Then (6.2) reduces to

$$J_{\mathbf{g}}^T \mathbf{y} = \mathbf{0}$$

Since $J_{\mathbf{g}}$ has rank m, this has only the trivial solution $\mathbf{y} = \mathbf{0}$ (Section 3.6.1). We have shown that the system (6.1) has only the trivial solution $(\mathbf{0}, \mathbf{0})$. This implies that the matrix J is nonsingular.

6.2 The Lagrangean for this problem is

$$L = f(\mathbf{x}) - \boldsymbol{\lambda}^T\big(\mathbf{g}(\mathbf{x}) - \mathbf{c}\big)$$

By Corollary 6.1.1

$$\nabla v(\mathbf{c}) = D_{\mathbf{c}} L = \boldsymbol{\lambda}$$

6.3 Optimality implies

$$f(\mathbf{x}^1, \boldsymbol{\theta}^1) \geq f(\mathbf{x}, \boldsymbol{\theta}^1) \text{ and } f(\mathbf{x}^2, \boldsymbol{\theta}^2) \geq f(\mathbf{x}, \boldsymbol{\theta}^2) \text{ for every } \mathbf{x} \in X$$

In particular

$$f(\mathbf{x}^1, \boldsymbol{\theta}^1) \geq f(\mathbf{x}^2, \boldsymbol{\theta}^1) \text{ and } f(\mathbf{x}^2, \boldsymbol{\theta}^2) \geq f(\mathbf{x}^1, \boldsymbol{\theta}^2)$$

Adding these inequalities

$$f(\mathbf{x}^1, \boldsymbol{\theta}^1) + f(\mathbf{x}^2, \boldsymbol{\theta}^2) \geq f(\mathbf{x}^2, \boldsymbol{\theta}^1) + f(\mathbf{x}^1, \boldsymbol{\theta}^2)$$

Rearranging and using the bilinearity of f gives

$$f(\mathbf{x}^1 - \mathbf{x}^2, \boldsymbol{\theta}^1) \geq f(\mathbf{x}^1 - \mathbf{x}^2, \boldsymbol{\theta}^2)$$

and

$$f(\mathbf{x}^1 - \mathbf{x}^2, \boldsymbol{\theta}^1 - \boldsymbol{\theta}^2) \geq 0$$

6.4 Let p_1 denote the profit maximizing price with the cost function $c_1(y)$ and let y_1 be the corresponding output. Similarly let p_2 and y_2 be the profit maximizing price and output when the costs are given by $c_2(y)$.

With cost function c_1, the firms profit is

$$\Pi = py - c_1(y)$$

Since this is maximised at p_1 and y_1 (although the monopolist could have sold y_2 at price p_2)

$$p_1 y_1 - c_1(y_1) \geq p_2 y_2 - c_1(y_2)$$

Rearranging

$$p_1 y_1 - p_2 y_2 \geq c_1(y_1) - c_1(y_2) \tag{6.4}$$

The increase in revenue in moving from y_2 to y_1 is greater than the increase in cost.

Similarly

$$p_2 y_2 - c_2(y_2) \geq p_1 y_1 - c_2(y_1)$$

which can be rearranged to yield

$$c_2(y_1) - c_2(y_2) \geq p_1 y_1 - p_2 y_2$$

Combining the previous inequality with (6.4) yields

$$c_2(y_1) - c_2(y_2) \geq c_1(y_1) - c_1(y_2) \tag{6.5}$$

6.5 By Theorem 6.2

$$D_{\mathbf{w}} \Pi[\mathbf{w}, p] = -\mathbf{x}^* \text{ and } D_p \Pi[\mathbf{w}, p] = y^*$$

and therefore

$$\begin{aligned}
D_p y(p, \mathbf{w}) &= D^2_{pp} \Pi(p, \mathbf{w}) \geq 0 \\
D_{w_i} x_i(p, \mathbf{w}) &= -D^2_{w_i w_i} \Pi(p, \mathbf{w}) \leq 0 \\
D_{w_j} x_i(p, \mathbf{w}) &= -D^2_{w_i w_j} \Pi(p, \mathbf{w}) = D_{w_i} x_j(p, \mathbf{w}) \\
D_p x_i(p, \mathbf{w}) &= -D^2_{w_i p} \Pi(p, \mathbf{w}) = -D_{w_i} y(p, \mathbf{w})
\end{aligned}$$

since Π is convex and therefore $H_\Pi(\mathbf{w}, p)$ is symmetric (Theorem 4.2) and nonnegative definite (Proposition 4.1).

6.6 By Shephard's lemma (6.17)

$$x_i(w, y) = D_{w_i} c(w, y)$$

Using Young's theorem (Theorem 4.2),

$$\begin{aligned} D_y x_i[\mathbf{w}, y] &= D^2_{w_i y} c[\mathbf{w}, y] \\ &= D^2_{y w_i} c[\mathbf{w}, y] \\ &= D_{w_i} D_y c[\mathbf{w}, y] \end{aligned}$$

Therefore

$$D_y x_i[\mathbf{w}, y] \geq 0 \iff D_{w_i} D_y c[\mathbf{w}, y] \geq 0$$

6.7 The demand functions must satisfy the budget contraint identically, that is

$$\sum_{i=1}^n p_i x_i(\mathbf{p}, m) = m \text{ for every } \mathbf{p} \text{ and } m$$

Differentiating with respect to m

$$\sum_{i=1}^n p_i D_m x_i[\mathbf{p}, m] = 1$$

This is the Engel aggregation condition, which simply states that any additional income be spent on some goods. Multiplying each term by $x_i m/(x_i m)$

$$\sum_{i=1}^n \frac{p_i x_i}{m} \frac{m}{x_i(\mathbf{p}, m)} D_m x_i[\mathbf{p}, m] = 1$$

the Engel aggregation condition can be written in elasticity form

$$\sum_{i=1}^n \alpha_i \eta_i = 1$$

where $\alpha_i = p_i x_i/m$ is the budget share of good i. On average, goods must have unit income elasticities.

Differentiating the budget constraint with respect to p_j

$$\sum_{i=1}^n p_i D_{p_j} x_i[\mathbf{p}, m] + x_j(p, m) = 0$$

This is the Cournot aggregation condition, which implies that an increase in the price of p_j is equivalent to a decrease in real income of $x_j dp_j$. Multiplying each term in the sum by x_i/x_i gives

$$\sum_{i=1}^n \frac{p_i x_i}{x_i} D_{p_j} x_i[\mathbf{p}, m] = -x_j$$

Multiplying through by p_j/m

$$\sum_{i=1}^n \frac{p_i x_i}{m} \frac{p_j}{x_i} D_{p_j} x_i[\mathbf{p}, m] = -\frac{p_j x_j}{m}$$

$$\sum_{i=1}^n \alpha_i \epsilon_{ij} = -\alpha_j$$

6.8 Supermodularity of $\Pi(\mathbf{x}, p, -\mathbf{w})$ follows from Exercises 2.50 and 2.51. To show strictly increasing differences, consider two price vectors $\mathbf{w}^2 \geq \mathbf{w}^1$

$$\begin{aligned}\Pi(\mathbf{x}, p, -\mathbf{w}^1) - \Pi(\mathbf{x}, p, -\mathbf{w}^2) &= \sum_{i=1}^n (-w_i^1)x_i - \sum_{i=1}^n (-w_i^2)x_i \\ &= \sum_{i=1}^n (w_i^2 - w_i^1)x_i\end{aligned}$$

Since $\mathbf{w}^2 \geq \mathbf{w}^1$, $\mathbf{w}^2 - \mathbf{w}^1 \geq \mathbf{0}$ and $\sum_{i=1}^n (w_i^2 - w_i^1)x_i$ is strictly increasing in $\mathbf{x}$.

6.9 For any $p^2 \geq p^1$, $y^2 = f(p^2) \leq f(p^1) = y^1$ and $c(y^1, \theta) - c(y^2, \theta)$ is increasing in θ and therefore $-\big(c(f(p^2), \theta) - c(f(p^1), \theta)\big)$ is increasing in θ.

6.10 The firm's optimization problem is

$$\max_{y \in \Re_+} \theta p y - c(y)$$

The objective function

$$f(y, p, \theta) = \theta p y - c(y)$$

is

- supermodular in y (Exercise 2.49)
- displays strictly increasing differences in (y, θ) since

$$f(y^2, p, \theta) - f(y^1, p, \theta) = \theta p(y^2 - y^1) - \big(c(y^2) - c(y^1)\big)$$

  is strictly increasing in θ for $y^2 > y^1$.

Therefore (Corollary 2.1.2), the firm's output correspondence is strongly increasing and every selection is increasing (Exercise 2.45). Therefore, the firm's output increases as the yield increases. It is analogous to an increase in the exogenous price.

6.11 With two factors, the Hessian is

$$H_f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

Therefore, its inverse is (Exercise 3.104)

$$H_f^{-1} = \frac{1}{\Delta}\begin{pmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{pmatrix}$$

where $\Delta = f_{11}f_{22} - f_{12}f_{21} \geq 0$ by the second-order condition. Therefore, the Jacobian of the demand functions is

$$\begin{pmatrix} D_{w_1}x_1 & D_{w_2}x_1 \\ D_{w_1}x_2 & D_{w_2}x_2 \end{pmatrix} = \frac{1}{p}H_f^{-1} = \frac{1}{p\Delta}\begin{pmatrix} f_{22} & -f_{12} \\ -f_{21} & f_{11} \end{pmatrix}$$

Therefore

$$D_{w_1}x_2 = -\frac{f_{21}}{p\Delta}\begin{cases} < 0 & \text{if } f_{21} > 0 \\ \geq 0 & \text{otherwise} \end{cases}$$