Chapter 2: Functions

2.1 In general, the birthday mapping is not one-to-one since two individuals may have the same birthday. It is not onto since some days may be no one's birthday.

2.2 The origin $\mathbf{0}$ is fixed point for every θ. Furthermore, when $\theta = 0$, f is an identity function and every point is a fixed point.

2.3 For every $x \in X$, there exists some $y \in Y$ such that $f(x) = y$, whence $x \in f^{-1}(y)$. Therefore, every x belongs to some contour. To show that distinct contours are disjoint, assume $x \in f^{-1}(y_1) \cap f^{-1}(y_2)$. Then $f(x) = y_1$ and also $f(x) = y_2$. Since f is a function, this implies that $y_1 = y_2$.

2.4 Assume f is one-to-one and onto. Then, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. That is, $f^{-1}(y) \neq \emptyset$ for every $y \in Y$. If f is one to one, $f(x) = y = f(x')$ implies $x = x'$. Therefore, $f^{-1}(y)$ consists of a single element. Therefore, the inverse function f^{-1} exists.

Conversely, assume that $f\colon X \to Y$ has an inverse f^{-1}. As f^{-1} is a function mapping Y to X, it must be defined for every $y \in Y$. Therefore f is onto. Assume there exists $x, x' \in X$ and $y \in Y$ such that $f(x) = y = f(x')$. Then $f^{-1}(y) = x$ and also $f^{-1}(y) = x'$. Since f^{-1} is a function, this implies that $x = x'$. Therefore f is one-to-one.

2.5 Choose any $x \in X$ and let $y = f(x)$. Since f is one-to-one, $x = f^{-1}(y) = f^{-1}(f(x))$. The second identity is proved similarly.

2.6 (2.2) implies for every $x \in \Re$

$$e^x e^{-x} = e^0 = 1$$

and therefore

$$e^{-x} = \frac{1}{e^x} \tag{2.1}$$

For every $x \geq 0$

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots > 0$$

and therefore by (2.1) $e^x > 0$ for every $x \in \Re$. For every $x \geq 1$

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \geq 1 + x \to \infty \text{ as } x \to \infty$$

and therefore $e^x \to \infty$ as $x \to \infty$. By (2.1) $e^x \to 0$ as $x \to -\infty$.

2.7

$$\begin{aligned} \frac{e^x}{x} &= \frac{e^{x/2} e^{x/2}}{2\frac{x}{2}} \\ &= \frac{1}{2} \left(\frac{e^{x/2}}{x/2} \right) e^{x/2} \to \infty \text{ as } x \to \infty \end{aligned}$$

since the term in brackets is strictly greater than 1 for any $x > 0$. Similarly

$$\begin{aligned}\frac{e^x}{x} &= \frac{(e^{x/(n+1)})^n e^{x/(n+1)}}{(n+1)^n(\frac{x}{n+1})^n} \\ &= \frac{1}{(n+1)^n}\left(\frac{e^{x/(n+1)}}{x/(n+1)}\right)^n e^{x/(n+1)} \to \infty\end{aligned}$$

2.8 Assume that $S \subseteq \Re$ is compact. Then S is bounded (Proposition 1.1), and there exists M such that $|x| \leq M$ for every $x \in S$. For all $n \geq m \geq 2M$

$$\begin{aligned}|f_n(x) - f_m(x)| = \left|\sum_{k=m+1}^{n} \frac{x^k}{k!}\right| &\leq \left|\frac{x^{m+1}}{(m+1)!}\sum_{k=0}^{n-m}\left(\frac{x}{m}\right)^k\right| \\ &\leq \left|\frac{M^{m+1}}{(m+1)!}\sum_{k=0}^{n-m}\left(\frac{M}{m}\right)^k\right| \\ &\leq \frac{M^{m+1}}{(m+1)!}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-m}\right) \\ &\leq 2\frac{M^{m+1}}{(m+1)!} \leq 2\left(\frac{M}{m+1}\right)^{m+1} \leq \left(\frac{1}{2}\right)^m\end{aligned}$$

by Exercise 1.206. Therefore f_n converges to f for all $x \in S$.

2.9 This is a special case of Example 2.8. For any $f, g \in F(X)$, define

$$\begin{aligned}(f + g) &= f(x) + g(x) \\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

With these definitions $f + g$ and αf also map X to $\Re$. Hence $F(X)$ is closed under addition and scalar multiplication. It is straightforward but tedious to verify that $F(X)$ satisfies the other requirements of a linear space.

2.10 The zero element in $F(X)$ is the constant function $f(x) = 0$ for every $x \in X$.

2.11
1. From the definition of $\|f\|$ it is clear that
   - $\|f\| \geq 0$.
   - $\|f\| = 0$ of and only f is the zero functional.
   - $\|\alpha f\| = |\alpha|\ \|f\|$ since $\sup_{x\in X} |\alpha f(x)| = |\alpha| \sup_{x\in X} |f(x)|$

   It remains to verify the triangle inequality, namely

$$\begin{aligned}\|f + g\| &= \sup_{x\in X} |(f + g)(x)| \\ &= \sup_{x\in X} |f(x) + g(x)| \\ &\leq \sup_{x\in X} \{\ |f(x)| + |g(x)|\ \} \\ &\leq \sup_{x\in X} |(f(x)| + \sup_{x\in X} |g(x)| \\ &= \|f\| + \|g\|\end{aligned}$$

2. Consequently, for any $f \in B(X)$, $\alpha f(x) \leq |\alpha|\ \|f\|$ for every $x \in X$ and therefore $\alpha f \in B(X)$. Similarly, for any $f, g \in B(X)$, $(f + g)(x) \leq \|f\| + \|g\|$ for every

$x \in X$ and therefore $f + g \in B(X)$. Hence, $B(X)$ is closed under addition and scalar multiplication; it is a subspace of the linear space $F(X)$. We conclude that $B(X)$ is a normed linear space.

3. To show that $B(X)$ is complete, assume that (f^n) is a Cauchy sequence in $B(X)$. For every $x \in X$

$$|f^n(x) - f^m(x)| \leq \|f^n - f^m\| \to 0$$

Therefore, for $x \in X$, $f^n(x)$ is a Cauchy sequence of real numbers. Since $\Re$ is complete, this sequence converges. Define the function

$$f(x) = \lim_{n \to \infty} f^n(x)$$

We need to show

- $\|f^n - f\| \to 0$ and
- $f \in B(X)$

(f^n) is a Cauchy sequence. For given $\epsilon > 0$, choose N such that $\|f^n - f^m\| < \epsilon/2$ for very $m, n \geq N$. For any $x \in X$ and $n \geq N$,

$$\begin{aligned} |f^n(x) - f(x)| &\leq |f^n(x) - f^m(x)| + |f^m(x) - f(x)| \\ &\leq \|f^n - f^m\| + |f^m(x) - f(x)| \end{aligned}$$

By suitable choice of m (which may depend upon x), each term on the right can be made smaller than $\epsilon/2$ and therefore

$$|f^n(x) - f(x)| < \epsilon$$

for every $x \in X$ and $n \geq N$.

$$\|f^n - f\| = \sup_{x \in X} |f^n(x) - f(x)| \leq \epsilon$$

Finally, this implies $\|f\| = \lim_{n \to \infty} \|f^n\|$. Therefore $f \in B(X)$.

2.12 If the die is fair, the probability of the elementary outcomes is

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = 1/6$$

By Condition 3

$$P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/2$$

2.13 The profit maximization problem of a competitive single-output firm is to choose the combination of inputs $\mathbf{x} \in \Re^n_+$ and scale of production y to maximize net profit. This is summarized in the constrained maximization problem

$$\max_{\mathbf{x}, y} py - \sum_{i=1}^n w_i x_i$$
$$\text{subject to } \mathbf{x} \in V(y)$$

where py is total revenue and $\sum_{i=1}^n w_i x_i$ total cost. The profit function, which depends upon both p and $\mathbf{w}$, is defined by

$$\Pi(p, \mathbf{w}) = \max_{y, \mathbf{x} \in V(y)} py - \sum_{i=1}^n w_i x_i$$

For analysis, it is convenient to represent the technology $V(y)$ by a production function (Example 2.24). The firm's optimization can then be expressed as

$$\max_{\mathbf{x}\in\Re^n_+} pf(\mathbf{x}) - \sum_{i=1}^n w_i x_i$$

and the profit function as

$$\Pi(p,\mathbf{w}) = \max_{\mathbf{x}\in\Re^n_+} pf(\mathbf{x}) - \sum_{i=1}^n w_i x_i$$

2.14 1. Assume that production is profitable at $\mathbf{p}$. That is, there exists some $\mathbf{y} \in Y$ such that $f(\mathbf{y},\mathbf{p}) > 0$. Since the technology exhibits constant returns to scale, Y is a cone (Example 1.101). Therefore $\alpha\mathbf{y} \in Y$ for every $\alpha > 0$ and

$$f(\alpha\mathbf{y},\mathbf{p}) = \sum_i p_i(\alpha y_i) = \alpha \sum_i p_i y_i = \alpha f(\mathbf{y},\mathbf{p})$$

Therefore $\{\, f(\alpha\mathbf{y},\mathbf{p}) : \alpha > 0 \,\}$ is unbounded and

$$\Pi(\mathbf{p}) = \sup_{\mathbf{y}\in Y} f(\mathbf{y},\mathbf{p}) \geq \sup_{\alpha>0} f(\alpha\mathbf{y},\mathbf{p}) = +\infty$$

2. Assume to the contrary that there exists $\mathbf{p} \in \Re^n_+$ with $\Pi(\mathbf{p}) = \pi \notin \{\,0,+\infty,-\infty\,\}$. There are two possible cases.

   (a) $0 < \pi < +\infty$. Since $\pi = \sup_{y\in Y} f(\mathbf{y},\mathbf{p}) > 0$, there exists $\mathbf{y} \in Y$ such that $f(\mathbf{y},\mathbf{p}) > 0$ The previous part implies $\Pi(\mathbf{p}) = +\infty$.

   (b) $-\infty < \pi < 0$. Then there exists $\mathbf{y} \in Y$ such that $f(\mathbf{y},\mathbf{p}) < 0$ By a similar argument to the previous part, this implies $\Pi(\mathbf{p}) = -\infty$.

2.15 Assume $\mathbf{x}^*$ is a solution to (2.4).

$$f(\mathbf{x}^*,\boldsymbol{\theta}) \geq f(\mathbf{x},\boldsymbol{\theta}) \text{ for every } \mathbf{x} \in G(\boldsymbol{\theta})$$

and therefore

$$f(\mathbf{x}^*,\boldsymbol{\theta}) \geq \sup_{\mathbf{x}\in G(\boldsymbol{\theta})} f(\mathbf{x},\boldsymbol{\theta}) = v(\boldsymbol{\theta})$$

On the other hand $\mathbf{x}^* \in G(\boldsymbol{\theta})$ and therefore

$$v(\boldsymbol{\theta}) = \sup_{\mathbf{x}\in G(\boldsymbol{\theta})} f(\mathbf{x},\boldsymbol{\theta}) \geq f(\mathbf{x}^*,\boldsymbol{\theta})$$

Therefore, $\mathbf{x}^*$ satisfies (2.5). Conversely, assume $\mathbf{x}^* \in G(\boldsymbol{\theta})$ satisfies (2.5). Then

$$f(\mathbf{x}^*,\boldsymbol{\theta}) = v(\boldsymbol{\theta}) = \sup_{\mathbf{x}\in G(\boldsymbol{\theta})} f(\mathbf{x},\boldsymbol{\theta}) \geq f(\mathbf{x},\boldsymbol{\theta}) \text{ for every } \mathbf{x} \in G(\boldsymbol{\theta})$$

$\mathbf{x}^*$ solve (2.4).

2.16 The assumption that $G(x) \neq \emptyset$ for every $x \in X$ implies $\Gamma(x_0) \neq \emptyset$ for every $x_0 \in X$. There always exist feasible plans from any starting point. Since u is bounded, there exists M such that $|f(x_t,x_{t+1})| \leq M$ for every $\mathbf{x} \in \Gamma(x_0)$. Consequently, for every $\mathbf{x} \in \Gamma(x_0)$, $U(\mathbf{x}) \in \Re$ and

$$|U(\mathbf{x})| = \left|\sum_{t=0}^\infty \beta^t f(x_t,x_{t+1})\right| \leq \sum_{t=0}^\infty \beta^t \,|f(x_t,x_{t+1})| \leq \sum_{t=0}^\infty \beta^t M = \frac{M}{1-\beta}$$

using the formula for a geometric series (Exercise 1.108). Therefore

$$v(x_0) = \sup_{\mathbf{x} \in \Gamma(x_0)} U(\mathbf{x}) \leq \frac{M}{1-\beta}$$

and $v \in B(X)$. Next, we note that for every feasible plan $\mathbf{x} \in \Gamma(x_0)$

$$\begin{aligned} U(\mathbf{x}) &= \sum_{t=0}^{\infty} \beta^t f(x_t, x_{t+1}) \\ &= f(x_0, x_1) + \beta \sum_{t=0}^{\infty} \beta^t f(x_{t+1}, x_{t+2}) \\ &= f(x_0, x_1) + \beta U(\mathbf{x}') \end{aligned} \tag{2.2}$$

where $\mathbf{x}' = (x_1, x_2, \dots)$ is the continuation of the plan $\mathbf{x}$ starting at x_1. For any $x_0 \in X$ and $\epsilon > 0$, there exists a feasible plan $\mathbf{x} \in \Gamma(x_0)$ such that

$$U(\mathbf{x}) \geq v(x_0) - \epsilon$$

Let $\mathbf{x}' = (x_1, x_2, \dots)$ be the continuation of the plan $\mathbf{x}$ starting at x_1. Using (2.2) and the fact that $U(\mathbf{x}') \leq v(x_1)$, we conclude that

$$\begin{aligned} v(x_0) - \epsilon &\leq U(\mathbf{x}) \\ &= f(x_0, x_1) + \beta U(\mathbf{x}') \\ &\leq f(x_0, x_1) + \beta v(x_1) \\ &\leq \sup_{y \in G(x)} \{ f(x_0, y) + \beta v(y) \} \end{aligned}$$

Since this is true for every $\epsilon > 0$, we must have

$$v(x_0) \leq \sup_{y \in G(x)} \{ f(x_0, y) + \beta v(y) \} \tag{2.3}$$

On the other hand, choose any $x_1 \in G(x_0) \subseteq X$. Since

$$v(x_1) = \sup_{\mathbf{x} \in \Gamma(x_1)} U(\mathbf{x})$$

there exists a feasible plan $\mathbf{x}' = (x_1, x_2, \dots)$ starting at x_1 such that

$$U(\mathbf{x}') \geq v(x_1) - \epsilon$$

Moreover, the plan $\mathbf{x} = (x_0, x_1, x_2, \dots)$ is feasible from x_0 and

$$v(x_0) \geq U(\mathbf{x}) = f(x_0, x_1) + \beta U(\mathbf{x}') \geq f(x_0, x_1) + \beta v(x_1) - \beta\epsilon$$

Since this is true for every $\epsilon > 0$ and $x_1 \in G(x_0)$, we conclude that

$$v(x_0) \geq \sup_{y \in G(x)} \{ f(x_0, y) + \beta v(y) \}$$

Together with (2.3) this establishes the required result, namely

$$v(x_0) = \sup_{y \in G(x)} \{ f(x_0, y) + \beta v(y) \}$$

2.17 Assume $\mathbf{x}$ is optimal, so that

$$U(\mathbf{x}^*) \geq U(\mathbf{x}) \text{ for every } \mathbf{x} \in \Gamma(x_0)$$

This implies (using (2.2))

$$f(x_0, x_1^*) + \beta U(\mathbf{x}^{*\prime}) \geq f(x_0, x_1) + \beta U(\mathbf{x}')$$

where $\mathbf{x}' = (x_1, x_2, \dots)$ is the continuation of the plan $\mathbf{x}$ starting at x_1 and $\mathbf{x}^{*\prime} = (x_1^*, x_2^*, \dots)$ is the continuation of the plan $\mathbf{x}^*$. In particular, this is true for every plan $\mathbf{x} \in \Gamma(x_0)$ with $x_1 = x_1^*$ and therefore

$$f(x_0, x_1^*) + \beta U(\mathbf{x}^{*\prime}) \geq f(x_0, x_1^*) + \beta U(\mathbf{x}') \text{ for every } \mathbf{x}' \in \Gamma(x_1^*)$$

which implies that

$$U(\mathbf{x}^{*\prime}) \geq U(\mathbf{x}') \text{ for every } \mathbf{x}' \in \Gamma(x_1^*)$$

That is, $\mathbf{x}^{*\prime}$ is optimal starting at x_1^* and therefore $U(\mathbf{x}^{*\prime}) = v(x_1^*)$ (Exercise 2.15). Consequently

$$v(x_0) = U(\mathbf{x}^*) = f(x_0, x_1^*) + \beta U(\mathbf{x}^{*\prime}) = f(x_0, x_1^*) + \beta v(x_1^*)$$

This verifies (2.13) for $t = 0$. A similar argument verifies (2.13) for any period t.

To show the converse, assume that $\mathbf{x}^* = (x_0, x_1^*, x_2^*, \dots) \in \Gamma(x_0)$ satisfies (2.13). Successively using (2.13)

$$\begin{aligned}
v(x_0) &= f(x_0, x_1^*) + \beta v(x_1^*) \\
&= f(x_0, x_1^*) + \beta f(x_1^*, x_2^*) + \beta^2 v(x_1^*) \\
&= \sum_{t=0}^{1} \beta^t f(x_t^*, x_{t+1}^*) + \beta^2 v(x_2^*) \\
&= \sum_{t=0}^{2} \beta^t f(x_t^*, x_{t+1}^*) + \beta^3 v(x_3^*) \\
&= \sum_{t=0}^{T-1} \beta^t f(x_t, xt+1) + \beta^T v(x_T^*)
\end{aligned}$$

for any $T = 1, 2, \dots$. Since v is bounded (Exercise 2.16), $\beta^T v(x_T^*) \to 0$ as $T \to \infty$ and therefore

$$v(x_0) = \sum_{t=0}^{\infty} \beta^t f(x_t, x_{t+1}) = U(\mathbf{x}^*)$$

Again using Exercise 2.15, $\mathbf{x}^*$ is optimal.

2.18 We have to show that

- for any $v \in B(X)$, Tv is a functional on X.
- Tv is bounded.

Since $F \in B(X \times X)$, there exists $M_1 < \infty$ such that $|f(x,y)| \leq M_1$ for every $(x,y) \in X \times X$. Similarly, for any $v \in B(X)$, there exists $M_2 < \infty$ such that $|v(x)| \leq M_2$ for every $x \in X$. Consequently for every $(x,y) \in X \times X$ and $v \in B(X)$

$$|f(x,y) + \beta v(y)| \leq |f(x,y)| + \beta\, |v(y)| \leq M_1 + \beta M_2 < \infty \tag{2.4}$$

For each $x \in X$, the set

$$S_x = \{\, f(x,y) + \beta v(y) : y \in G(x) \,\}$$

is a nonempty bounded subset of $\Re$, which has least upper bound. Therefore

$$(Tv)(x) = \sup S_x = \sup_{y \in G(x)} f(x,y) + \beta v(y)$$

defines a functional on X. Moreover by (2.4)

$$|(Tv)(k)| \leq M_1 + \beta M_2 < \infty$$

Therefore $Tv \in B(X)$.

2.19 Let $N = \{1, 2, 3\}$. Any individual is powerless so that

$$w(\{i\}) = 0 \qquad i = 1, 2, 3$$

Any two players can allocate the \$1 to between themselves, leaving the other player out. Therefore

$$w(\{i, j\}) = 1 \qquad i, j \in N, i \neq j$$

The best that the three players can achieve is to allocate the \$1 amongst themselves, so that

$$w(N) = 1$$

2.20 If the consumers preferences are continuous and strictly convex, she has a unique optimal choice $\mathbf{x}^*$ for every set of prices $\mathbf{p}$ and income m in P (Example 1.116). Therefore, the demand correspondence is single valued.

2.21 Assume $s_i^* \in B(\mathbf{s}^*)$ for every $i \in N$. Then for every player $i \in N$

$$(s_i, \mathbf{s}_{-i}) \succsim_i (s_i', \mathbf{s}_{-i}) \text{ for every } s_i' \in S_i$$

$\mathbf{s}^* = (s_1^*, s_2^*, \dots, s_n^*)$ is a Nash equilibrium. Conversely, assume $\mathbf{s}^* = (s_1^*, s_2^*, \dots, s_n^*)$ is a Nash equilibrium. Then for every player $i \in N$

$$(s_i, \mathbf{s}_{-i}) \succsim_i (s_i', \mathbf{s}_{-i}) \text{ for every } s_i' \in S_i$$

which implies that

$$s_i^* \in B(\mathbf{s}^*) \text{ for every } i \in N$$

2.22 For any nonempty compact set $T \subseteq S$, $B(T)$ is nonempty and compact provided $\succsim_i$ is continuous (Proposition 1.5) and $B(T) \subseteq T$. Therefore

$$B_i^1 \supseteq B_i^2 \supseteq B_i^3 \dots$$

is a nested sequence of nonempty compact sets. By the nested intersection theorem (Exercise 1.117), $R_i = \bigcap_{n=0}^{\infty} B_i^n \neq \emptyset$.

2.23 If $\mathbf{s}^*$ is a Nash equilibrium, $s_i \in B_i^n$ for every n.

2.24 For any $\boldsymbol{\theta}$, let $\mathbf{x}^* \in \varphi(\boldsymbol{\theta})$. Then

$$f(\mathbf{x}^*, \boldsymbol{\theta}) \geq f(\mathbf{x}, \boldsymbol{\theta}) \qquad \text{for every } \ \mathbf{x} \in G(\boldsymbol{\theta})$$

Therefore

$$f(\mathbf{x}^*, \boldsymbol{\theta}) \geq v(\boldsymbol{\theta}) = \sup_{\mathbf{x} \in G(\boldsymbol{\theta})} f(\mathbf{x}, \boldsymbol{\theta})$$

Conversely

$$v(\boldsymbol{\theta}) = \sup_{\mathbf{x} \in G(\boldsymbol{\theta})} f(\mathbf{x}, \boldsymbol{\theta}) \geq \sup_{\mathbf{x} \in G(\boldsymbol{\theta})} f(\mathbf{x}, \boldsymbol{\theta}) \geq f(\mathbf{x}^*, \boldsymbol{\theta}) \text{ for every } \ \mathbf{x}^* \in \varphi(\boldsymbol{\theta})$$

Consequently

$$v(\boldsymbol{\theta}) = f(\mathbf{x}^*, \boldsymbol{\theta}) \text{ for any } \mathbf{x}^* \in \varphi(\boldsymbol{\theta})$$

2.25 The graph of V is

$$\text{graph}(V) = \{\, (y, \mathbf{x}) \in \Re_+ \times \Re^n_+ : \mathbf{x} \in V(y) \,\}$$

while the production possibility set Y is

$$Y = \{\, (y, -\mathbf{x}) \in \Re_+ \times \Re^n_+ : x \in V(y) \,\}$$

Assume that Y is convex and let $(y^i, \mathbf{x}^i) \in \text{graph}(V)$, $i = 1, 2$. This means that

$$(y^1, -\mathbf{x}^1) \in Y \text{ and } \ (y^2, -\mathbf{x}^2) \in Y$$

Let

$$\bar{y} = \alpha y^1 + (1-\alpha) y^2 \text{ and } \ \bar{\mathbf{x}} = \alpha \mathbf{x}^1 + (1-\alpha)\mathbf{x}^2$$

for some $0 \leq \alpha \leq 1$. Since Y is convex

$$(\bar{y}, -\bar{\mathbf{x}}) = \alpha(y^1, -\mathbf{x}^1) + (1-\alpha)(y^2, -\mathbf{x}^2) \in Y$$

and therefore $\bar{\mathbf{x}} \in V(\bar{y})$ so that $(\bar{y}, \bar{\mathbf{x}}) \in \text{graph}(V)$. That is graph$(V)$ is convex.

Conversely, assuming graph(V) is convex, if $(y^i, -\mathbf{x}^i) \in Y$, $i = 1, 2$, then $(y^i, \mathbf{x}^i) \in$ graph(V) and therefore

$$(\bar{y}, \bar{\mathbf{x}}) \in \text{graph}(V) \implies \bar{\mathbf{x}} \in V(\bar{y}) \implies (\bar{y}, -\bar{\mathbf{x}}) \in Y$$

so that Y is convex.

2.26 The graph of φ is

$$\text{graph}(G) = \{\, (\boldsymbol{\theta}, \mathbf{x}) \in \Theta \times X : \mathbf{x} \in G(\boldsymbol{\theta}) \,\}$$

Assume that $(\boldsymbol{\theta}^i, \mathbf{x}^i) \in \text{graph}(G)$, $i = 1, 2$. This means that $\mathbf{x}^i \in G(\boldsymbol{\theta})$ and therefore $g^j(\mathbf{x}, \boldsymbol{\theta}) \leq c_j$ for every j and $i = 1, 2$. Since g^j is convex

$$g(\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2, \alpha\boldsymbol{\theta}^1 + (1-\alpha)\boldsymbol{\theta}^2) \geq \alpha g(\mathbf{x}^1, \boldsymbol{\theta}^1) + (1-\alpha) g(\mathbf{x}^2, \boldsymbol{\theta}^2) \geq c_j$$

Therefore $\alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2 \in G(\alpha\boldsymbol{\theta}^1 + (1-\alpha)\boldsymbol{\theta}^2)$ and $(\alpha\boldsymbol{\theta}^1 + (1-\alpha)\boldsymbol{\theta}^2, \alpha\mathbf{x}^1 + (1-\alpha)\mathbf{x}^2) \in$ graph(G). G is convex.

2.27 The identity function $I_X : X \to X$ is defined by $I_X(x) = x$ for every $x \in X$. Therefore

$$x_2 \succ_X x_1 \implies I_X(x_2) = x_2 \succ_X x_1 = I_X(x_1)$$

2.28 Assume that f and g are increasing. Then, for every $x_1, x_2 \in X$ with $x_2 \succsim_X x_1$

$$f(x_2) \succsim_Y f(x_1) \implies g(f(x_2)) \succsim_Z g(f(x_1))$$

$g \circ f$ is also increasing. Similarly, if f and g are strictly increasing,

$$x_2 \succ_X x_1 \implies f(x_2) \succ_Y f(x_1) \implies g(f(x_2)) \succ_Z g(f(x_1))$$

2.29 For every $y \in f(X)$, there exists a unique $x \in X$ such that $f(x) =$ y. (For if x_1, x_2 are such that $f(x_1) = f(x_2)$, then $x_1 = x_2$.) Therefore, f is one-to-one and onto $f(X)$, and so has an inverse (Exercise 2.4). Further

$$x_2 > x_2 \iff f(x_2) > f(x_1)$$

Therefore f^{-1} is strictly increasing.

2.30 Assume $f : X \to \Re$ is increasing. Then, for every $x_2 \succsim x_1$, $f(x_2) \geq f(x_1)$ which implies that $-f(x_2) \leq -f(x_1)$. $-f$ is decreasing.

2.31 For every $x_2 \succsim x_1$.

$$f(x_2) \geq f(x_1)$$
$$g(x_2) \geq g(x_1)$$

Adding

$$(f + g)(x_2) = f(x_2) + g(x_2) \geq f(x_1) + f(x_1) = (f + g)(x_1)$$

That is, $f + g$ is increasing. Similarly for every $\alpha \geq 0$

$$\alpha f(x_2) \geq \alpha f(x_1)$$

and therefore αf is increasing. By Exercise 1.186, the set of all increasing functionals is a convex cone in $F(X)$.

If f is strictly increasing, then for every $x_2 \succ x_1$,

$$f(x_2) > f(x_1)$$
$$g(x_2) \geq g(x_1)$$

Adding

$$(f + g)(x_2) = f(x_2) + g(x_2) > f(x_1) + g(x_1) = (f + g)(x_1)$$

$f + g$ is strictly increasing. Similarly for every $\alpha > 0$

$$\alpha f(x_2) > \alpha f(x_1)$$

αf is strictly increasing.

2.32 For every $x_2 \succ x_1$.

$$f(x_2) > f(x_1) > 0$$
$$g(x_2) > g(x_1) > 0$$

and therefore

$$(fg)(x_2) = f(x_2)g(x_2) > f(x_2)g(x_1) > f(x_1)g(x_1) = (fg)(x_1)$$

using Exercise 2.31.

2.33 By Exercise 2.31 and Example 2.53, each g_n is strictly increasing on $\Re_+$. That is

$$x_1 < x_2 \implies g_n(x_1) < g_n(x_2) \text{ for every } n \tag{2.5}$$

and therefore

$$\lim_{n\to\infty} g_n(x_1) \leq \lim_{n\to\infty} g_n(x_2)$$

This suffices to show that $g(x) = \lim_{n\to\infty} g_n(x)$ is increasing (not strictly increasing). However, $1 + x$ is strictly increasing, and therefore by Exercise 2.31

$$e^x = 1 + x + g(x)$$

is strictly increasing on $\Re_+$. While it is the case that $g = \lim g_n$ is strictly increasing on $\Re_+$, (2.5) does not suffice to show this.

2.34 For every $a > 0$, $a \log x$ is strictly increasing (Exercise 2.32) and therefore $e^{a \log x}$ is strictly increasing (Exercise 2.28). For every $a < 0$, $-a \log x$ is strictly increasing and therefore (Exercise 2.30 $a \log x$ is strictly decreasing. Therefore $e^{a \log x}$ is strictly decreasing (Exercise 2.28).

2.35 Apply Exercises 2.31 and 2.28 to Example 2.56.

2.36 u is (strictly) increasing so that

$$x_2 \succsim x_1 \implies u(x_2) \geq u(x_1)$$

To show the converse, assume that $x_1, x_2 \in X$ with $u(x_2) \geq u(x_1)$. Since $\succsim$ is complete, either $x_2 \succsim x_1$ or $x_1 \succ x_2$. However, the second possibility cannot occur since u is strictly increasing and therefore

$$x_1 \succ x_2 \implies u(x_1) > u(x_2)$$

contradicting the hypothesis that $u(x_2) \geq u(x_1)$. We conclude that

$$u(x_2) \geq u(x_1) \implies x_2 \succsim x_1$$

2.37 Assume u represents the preference ordering $\succsim$ on X and let $g\colon \Re \to \Re$ be strictly increasing. Then composition $g \circ u\colon X \to \Re$ is strictly increasing (Exercise 2.28). Therefore $g \circ u$ is a utility function (Example 2.58). Since g is strictly increasing

$$g(u(x_2)) \geq g(u(x_1)) \iff u(x_2) \geq u(x_1) \iff x_2 \succsim x_1$$

for every $x_1, x_2 \in X$ Therefore, $g \circ u$ also represents $\succsim$.

2.38
1. (a) Let $\bar{z} = \max_{i=1}^n x_i$. Then $\bar{\mathbf{z}} = \bar{z}\mathbf{1} \succsim \mathbf{x}$ and therefore $\bar{\mathbf{z}} \in Z_{\mathbf{x}}^+$. Similarly, let $\underline{z} = \min_{i=1}^n x_i$. Then $\underline{\mathbf{z}} = \underline{z}\mathbf{1} \in Z_{\mathbf{x}}^-$. Therefore, $Z_{\mathbf{x}}^+$ and $Z_{\mathbf{x}}^-$ are both nonempty. By continuity, the upper and lower contour sets $\succsim(\mathbf{x})$ and $\precsim(\mathbf{x})$ are closed. Z is a closed cone. Since

   $$Z_{\mathbf{x}}^+ = \succsim(\mathbf{x}) \cap Z \text{ and } Z_{\mathbf{x}}^- = \precsim(\mathbf{x}) \cap Z$$

   $Z_{\mathbf{x}}^+$ and $Z_{\mathbf{x}}^-$ are closed.

   (b) By completeness, $Z_{\mathbf{x}}^+ \cup Z_{\mathbf{x}}^- = Z$. Since Z is connected, $Z_{\mathbf{x}}^+ \cap Z_{\mathbf{x}}^- \neq \emptyset$. (Otherwise, Z is the union of two disjoint closed sets and hence the union of two disjoint open sets.)

   (c) Let $\mathbf{z}_{\mathbf{x}} \in Z_{\mathbf{x}}^+ \cap Z_{\mathbf{x}}^-$. Then $\mathbf{z} \succsim \mathbf{x}$ and also $\mathbf{z} \precsim \mathbf{x}$. That is, $\mathbf{z} \sim \mathbf{x}$.

(d) Suppose $\mathbf{x} \sim \mathbf{z}_{\mathbf{x}}^1$ and $\mathbf{x} \sim \mathbf{z}_{\mathbf{x}}^2$ with $\mathbf{z}_{\mathbf{x}}^1 \neq \mathbf{z}_{\mathbf{x}}^2$. Then either $\mathbf{z}_{\mathbf{x}}^1 > \mathbf{z}_{\mathbf{x}}^2$ or $\mathbf{z}_{\mathbf{x}}^1 < \mathbf{z}_{\mathbf{x}}^2$. Without loss of generality, assume $\mathbf{z}_{\mathbf{x}}^2 > \mathbf{z}_{\mathbf{x}}^1$. Then monotonicity and transitivity imply

$$\mathbf{x} \sim \mathbf{z}_{\mathbf{x}}^2 \succ \mathbf{z}_{\mathbf{x}}^1 \sim \mathbf{x}$$

which is a contradiction. Therefore $\mathbf{z}_{\mathbf{x}}$ is unique.

Let $z_{\mathbf{x}}$ denote the scale of $\mathbf{z}_{\mathbf{x}}$, that is $\mathbf{z}_{\mathbf{x}} = z_{\mathbf{x}}\mathbf{1}$. For every $\mathbf{x} \in \Re^n_+$, there is a unique $\mathbf{z}_{\mathbf{x}} \sim \mathbf{x}$ and the function $u\colon \Re^n_+ \to \Re$ given by $u(\mathbf{x}) = z_{\mathbf{x}}$ is well-defined. Moreover

$$\begin{aligned} \mathbf{x}_2 \succsim \mathbf{x}_1 &\iff \mathbf{z}_{\mathbf{x}_2} \succsim \mathbf{z}_{\mathbf{x}_1} \\ &\iff z_{\mathbf{x}_2} \geq z_{\mathbf{x}_1} \\ &\iff u(\mathbf{x}_2) \geq u(\mathbf{x}_1) \end{aligned}$$

u represents the preference order $\succsim$.

2.39 1. For every $x_1 \in \Re$, $(x_1, 2) \succ_L (x_1, 1)$ in the lexicographic order. If u represents $\succsim_L$, u is strictly increasing and therefore $u(x_1, 2) > u(x_1, 1)$. There exists a rational number $r(x_1)$ such that $u(x_1, 2) > r(x_1) > u(x_1, 1)$.

2. The preceding construction associates a rational number with every real number $x_1 \in \Re$. Hence r is a function from $\Re$ to the set of rational numbers Q. For any $x_1^1, x_1^2 \in \Re$ with $x_1^2 > x_1^1$

$$r(x_1^2) > u(x_1^2, 1) > u(x_1^1, 2) > r(x_1^1)$$

Therefore

$$x_1^2 > x_1^1 \implies r(x_1^2) > r(x_1^1)$$

r is strictly increasing.

3. By Exercise 2.29, r has an inverse. This implies that r is one-to-one and onto, which is impossible since Q is countable and $\Re$ is uncountable (Example 2.16). This contradiction establishes that $\succsim_L$ has no such representation u.

2.40 Let $\mathbf{a}^1, \mathbf{a}^2 \in A$ with $\mathbf{a}^1 \succsim_2 \mathbf{a}^2$. Since the game is strictly competitive, $\mathbf{a}^2 \succsim_1 \mathbf{a}^1$. Since u_1 represents $\succsim_1$, $u_1(\mathbf{a}^2) \geq u_1(\mathbf{a}^1)$ which implies that $-u_1(\mathbf{a}^2) \leq -u_1(\mathbf{a}^1)$, that is $u_2(\mathbf{a}^1) \geq u_2(\mathbf{a}^2)$ where $u_2 = -u_1$. Similarly

$$u_2(\mathbf{a}^1) \geq u_2(\mathbf{a}^2) \implies u_1(\mathbf{a}^1) \leq u_1(\mathbf{a}^2) \iff \mathbf{a}^1 \precsim_1 \mathbf{a}^2 \implies \mathbf{a}^1 \succsim_2 \mathbf{a}^2$$

Therefore $u_2 = -u_1$ represents $\succsim_2$ and

$$u_1(\mathbf{a}) + u_2(\mathbf{a}) = 0 \text{ for every } \mathbf{a} \in A$$

2.41 Assume $S \subsetneq T$. By superadditivity

$$w(T) \geq w(S) + w(T \setminus S) \geq w(S)$$

2.42 Assume $v, w \in B(X)$ with $w(y) \geq v(y)$ for every $y \in X$. Then for any $x \in X$

$$f(x, y) + \beta w(y) \geq f(x, y) + \beta v(y) \text{ for every } y \in X$$

and therefore

$$(Tw)(x) = \sup_{y \in G(x)} \{f(x, y) + \beta w(y)\} \geq \sup_{y \in G(x)} \{f(x, y) + \beta v(y)\} = (Tv)(x)$$

T is increasing.

2.43 For every $\theta_2 \geq \theta_1 \in \Theta$, if $x_1 \in G(\theta_1)$ and $x_2 \in G(\theta_2)$, then $x_1 \wedge x_2 \leq x_1$ and therefore $x_1 \wedge x_2 \in G(\theta_1)$. If $x_1 \geq x_2$, then $x_1 \vee x_2 = x_1 \leq g(\theta_1) \leq g(\theta_2)$ and therefore $x_1 \vee x_2 \in G(\theta_2)$. On the other hand, if $x_1 \leq x_2$, then $x_1 \vee x_2 = x_2 \in G(\theta_2)$.

2.44 Assume φ is increasing, and let $x_1, x_2 \in X$ with $x_2 \succsim x_1$. Let $y_1 \in \varphi(x_1)$. Choose any $y' \in \varphi(x_2)$. Since φ is increasing, $\varphi(x_2) \succsim_S \varphi(x_1)$ and therefore $y_2 = y_1 \vee y' \in \varphi(x_2)$. $y_2 \succsim y_1$ as required. Similarly, for every $y_2 \in \varphi(x_2)$, there exists some $y' \in \varphi(x_2)$ such that $y_1 = y' \wedge y_2 \in \varphi(x_1)$ with $y_2 \succsim y_1$.

2.45 Since $\varphi(x)$ is a sublattice, $\sup \varphi(x) \in \varphi(x)$ for every x. Therefore, the function

$$f(x) = \sup \varphi(x)$$

is a selection. Similarly

$$g(x) = \inf \varphi(x)$$

is a selection. Both f and g are increasing (Exercise 1.50).

2.46 Let x^1, x^2 belong to X with $x^2 \succsim x^1$. Choose $\mathbf{y}^1 = (y_1^1, y_2^1, \dots, y_n^1) \in \prod_i \varphi_i(x^1)$ and $\mathbf{y}^2 = (y_1^2, y_2^2, \dots, y_n^2) \in \prod_i \varphi_i(x^2)$. Then, for each $i = 1, 2, \dots, n$, $y_i^1 \in \varphi_i(x^1)$ and $y_i^2 \in \varphi_i(x^2)$. Since each φ_i is increasing, $y_i^1 \wedge y_i^2 \in \varphi_i(x^1)$ and $y_i^1 \vee y_i^2 \in \varphi_i(x^2)$ for each i. Therefore $\mathbf{y}^1 \wedge \mathbf{y}^2 \in \prod_i \varphi_i(x^1)$ and $\mathbf{y}^1 \vee \mathbf{y}^2 \in \prod_i \varphi_i(x^2)$. $\varphi(x) = \prod_i \varphi_i(x)$ is increasing.

2.47 Let x^1, x^2 belong to X with $x^2 \succsim x^1$. Choose $y^1 \in \bigcap_i \varphi_i(x^1)$ and $y^2 \in \bigcap_i \varphi_i(x^2)$. Then $y^1 \in \varphi_i(x^1)$ and $y^2 \in \varphi_i(x^2)$ for each $i = 1, 2, \dots, n$. Since each φ_i is increasing, $y^1 \wedge y^2 \in \varphi_i(x^1)$ and $y^1 \vee y^2 \in \varphi_i(x^2)$ for each i. Therefore $y^1 \wedge y^2 \in \bigcap_i \varphi_i(x^1)$ and $y^1 \vee y^2 \in \bigcap_i \varphi_i(x^2)$. $\varphi = \bigcap_i \varphi$ is increasing.

2.48 Let f be a selection from an always increasing correspondence $\varphi\colon X \rightrightarrows Y$. For every $x_1, x_2 \in X$, $f(x_1) \in \varphi(x_1)$ and $f(x_2) \in \varphi(x_2)$. Since φ is always increasing

$$x_1 \succsim_X x_2 \implies f(x_1) \succsim_Y f(x_2)$$

f is increasing. Conversely, assume every selection $f \in \varphi$ is increasing. Choose any $x_1, x_2 \in X$ with $x_1 \succsim x_2$. For every $y_1 \in \varphi(x_1)$ and $y_2 \in \varphi(x_2)$, there exists a selection f with $y_i = \varphi(x_i), i = 1, 2$. Since f is increasing,

$$x_1 \succsim_X x_2 \implies y_1 \succsim_Y y_2$$

φ is increasing.

2.49 Let $x_1, x_2 \in X$. If X is a chain, either $x_1 \succsim x_2$ or $x_2 \succsim x_1$. Without loss of generality , assume $x_2 \succsim x_1$. Then $x_1 \vee x_2 = x_2$ and $x_1 \wedge x_2 = x_1$ and (2.17) is satisfied as an identity.

2.50

$$\begin{aligned}
(f+g)(x_1 \vee x_2) + (f+g)(x_1 \wedge x_2) &= f(x_1 \vee x_2) + g(x_1 \vee x_2) + f(x_1 \wedge x_2) + g(x_1 \wedge x_2) \\
&= f(x_1 \vee x_2) + f(x_1 \wedge x_2) + g(x_1 \vee x_2) + g(x_1 \wedge x_2) \\
&\geq f(x_1) + f(x_2) + g(x_1) + g(x_2) \\
&= (f+g)(x_1) + (f+g)(x_2)
\end{aligned}$$

Similarly

$$f(x_1 \vee x_2) + f(x_1 \wedge x_2) \geq f(x_1) + f(x_2)$$

implies

$$\alpha f(x_1 \vee x_2) + \alpha f(x_1 \wedge x_2) \geq \alpha f(x_1) + \alpha f(x_2)$$

for all $\alpha \geq 0$. By Exercise 1.186, the set of all supermodular functions is a convex cone in $F(X)$.

2.51 Since f is supermodular and g is nonnegative definite,

$$\begin{aligned} f(x_1 \vee x_2)g(x_1 \vee x_2) &\geq \big(f(x_1) + f(x_2) - f(x_1 \wedge x_2)\big)g(x_1 \vee x_2) \\ &= f(x_2)g(x_1 \vee x_2) + \big(f(x_1) - f(x_1 \wedge x_2)\big)g(x_1 \vee x_2) \end{aligned}$$

for any $x_1, x_2 \in X$. Since f and g are increasing, this implies

$$f(x_1 \vee x_2)g(x_1 \vee x_2) \geq f(x_2)g(x_1 \vee x_2) + \big(f(x_1) - f(x_1 \wedge x_2)\big)g(x_1) \tag{2.6}$$

Similarly, since f is nonnegative definite, g supermodular, and f and g increasing

$$\begin{aligned} f(x_2)g(x_1 \vee x_2) &\geq f(x_2)\big(g(x_1) + g(x_2) - g(x_1 \wedge x_2)\big) \\ &= f(x_2)g(x_2) + f(x_2)\big(g(x_1) - g(x_1 \wedge x_2)\big) \\ &\geq f(x_2)g(x_2) + f(x_1 \wedge x_2)\big(g(x_1) - g(x_1 \wedge x_2)\big) \end{aligned}$$

Combining this inequality with (2.6) gives

$$\begin{aligned} f(x_1 \vee x_2)g(x_1 \vee x_2) &\geq f(x_2)g(x_2) + f(x_1 \wedge x_2)\big(g(x_1) - g(x_1 \wedge x_2)\big) \\ &\quad + \big(f(x_1) - f(x_1 \wedge x_2)\big)g(x_1) \\ &= f(x_2)g(x_2) + f(x_1 \wedge x_2)g(x_1) - f(x_1 \wedge x_2)g(x_1 \wedge x_2) \\ &\quad + f(x_1)g(x_1) - f(x_1 \wedge x_2)g(x_1) \\ &= f(x_2)g(x_2) - f(x_1 \wedge x_2)g(x_1 \wedge x2) + f(x_1)g(x_1) \end{aligned}$$

or

$$fg(x_1 \vee x_2) + fg(x_1 \wedge x_2) \geq fg(x_1) + fg(x_2)$$

fg is supermodular. (I acknowledge the help of Don Topkis in formulating this proof.)

2.52 Exercises 2.49 and 2.50.

2.53 For simplicity, assume that the firm produces two products. For every production plan $\mathbf{y} = (y_1, y_2)$,

$$\begin{aligned} \mathbf{y} &= (y_1, 0) \vee (0, y_2) \\ \mathbf{0} &= (y_1, 0) \wedge (0, y_2) \end{aligned}$$

If c is strictly submodular

$$c(\mathbf{w}, \mathbf{y}) + c(\mathbf{w}, \mathbf{0}) < c(\mathbf{w}, (y_1, 0)) + c(\mathbf{w}, (0, y_2))$$

Since $c(\mathbf{w}, \mathbf{0}) = 0$

$$c(\mathbf{w}, \mathbf{y}) < c(\mathbf{w}, (y_1, 0)) + c(\mathbf{w}, (0, y_2))$$

The technology displays economies of scope.

2.54 Assume (N, w) is convex, that is

$$w(S \cup T) + w(S \cap T) \geq w(S) + w(T) \text{ for every } \quad S, T \subseteq N$$

For all disjoint coalitions $S \cap T = \emptyset$

$$w(S \cup T) \geq w(S) + w(T)$$

w is superadditive.

2.55 Rewriting (2.18), this implies

$$w(S \cup T) - w(T) \geq w(S) - w(S \cap T) \text{ for every } \quad S, T \subseteq N \tag{2.7}$$

Let $S \subset T \subset N \setminus \{i\}$ and let $S' = S \cup \{i\}$. Substituting in (2.7)

$$w(S' \cup T) - w(T) \geq w(S') - w(S' \cap T)$$

Since $S \subset T$

$$\begin{aligned} S' \cup T &= (S \cup \{i\}) \cup T = T \cup \{i\} \\ S' \cap T &= (S \cup \{i\}) \cap T = S \end{aligned}$$

Substituting in the previous equation gives the required result, namely

$$w(T \cup \{i\}) - w(T) \geq w(S \cup \{i\}) - w(S)$$

Conversely, assume that

$$w(T \cup \{i\}) - w(T) \geq w(S \cup \{i\}) - w(S) \tag{2.8}$$

for every $S \subset T \subset N \setminus \{i\}$. Let S and T be arbitrary coalitions. Assume $S \cap T \subset S$ and $S \cap T \subset T$ (otherwise (2.18) is trivially satisfied). This implies that $T \setminus S \neq \emptyset$. Assume these players are labelled $1, 2, \dots, m$, that is $T \setminus S = \{1, 2, \dots, m\}$. By (2.8)

$$w(S \cup \{1\}) - w(S) \geq w((S \cap T) \cup \{1\}) - w(S \cap T) \tag{2.9}$$

Successively adding the remaining players in $T \setminus S$

$$\begin{aligned} w(S \cup \{1, 2\}) - w(S \cup \{1\}) &\geq w((S \cap T) \cup \{1, 2\}) - w((S \cap T) \cup \{1\}) \\ &\vdots \\ w(S \cup (T \setminus S)) - w(S \cup \{1, 2, \dots, m-1\}) &\geq w\big(S \cap T) \cup (T \setminus S)\big) \\ &\quad - w((S \cap T) \cup \{1, 2, \dots, m-1\}) \end{aligned}$$

Adding these inequalities to (2.9), we get

$$w(S \cup (T \setminus S)) - w(S) \geq w\big(S \cap T) \cup (T \setminus S)\big) - w(S \cap T)$$

This simplifies to

$$w(S \cup T) - w(S) \geq S(T) - w(S \cap T)$$

which can be arranged to give (2.18).

2.56 The cost allocation game is not convex. Let $S = \{AP, KM\}$, $T = \{KM, TN\}$. Then $S \cup T = \{AP, KM, TN\} = N$ and $S \cap T = \{KM\}$ and

$$w(S \cup T) + w(S \cap T) = 1530 < 1940 = 770 + 1170 = w(S) + w(T)$$

Alternatively, observe that TN's marginal contribution to coalition $\{KM, TN\}$ is 1170, which is greater than its marginal contribution to the grand coalition $\{AP, KM, TN\}$ $(1530 - 770 = 760)$.

2.57 f is supermodular if

$$f(x_1 \vee x_2) + f(x_1 \wedge x_2) \geq f(x_1) + f(x_2)$$

which can be rearranged to give

$$f(x_1 \vee x_2) - f(x_2) \geq f(x_1) - f(x_1 \wedge x_2)$$

If the right hand side of this inequality is nonnegative, then so *a fortiori* is the left hand side, that is

$$f(x_1) \geq f(x_1 \wedge x_2) \implies f(x_1 \vee x_2) \geq f(x_2)$$

If the right hand side is strictly positive, so must be the left hand side

$$f(x_1) > f(x_1 \wedge x_2) \implies f(x_1 \vee x_2) > f(x_2)$$

2.58 Assume $x_2 \succsim x_1 \in X$ and $y_2 \succsim_Y y_2 \in Y$. Assume that f displays increasing differences in (x, y), that is

$$f(x_2, y_2) - f(x_1, y_2) \geq f(x_2, y_1) - f(x_1, y_1) \tag{2.10}$$

Rearranging

$$f(x_2, y_2) - f(x_2, y_1) \geq f(x_1, y_2) - f(x_1, y_1) \tag{2.11}$$

Conversely, (2.11) implies (2.10) .

2.59 We showed in the text that supermodularity implies increasing differences. To show that reverse, assume that $f\colon X \times Y \to \Re$ displays increasing differences in (x, y). Choose any $(x_1, y_1), (x_2, y_2) \in X \times Y$. If $(x_1, y_1), (x_2, y_2)$ are comparable, so that either $(x_1, y_1) \succsim (x_2, y_2)$ or $(x_1, y_1) \precsim (x_2, y_2)$, then (2.17) holds has an equality. Therefore assume that $(x_1, y_1), (x_2, y_2)$ are incomparable. Without loss of generality, assume that $x_1 \precsim x_2$ while $y_1 \succsim y_2$. (This is where we require that X and Y be chains). This implies

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1, y_2) \text{ and } (x_1, y_1) \vee (x_2, y_2) = (x_2, y_1) \tag{2.12}$$

Increasing differences implies that

$$f(x_2, y_1) - f(x_1, y_1) \geq f(x_2, y_2) - f(x_1, y_2)$$

which can be rewritten as

$$f(x_2, y_1) + f(x_1, y_2) \geq f(x_1, y_1) + f(x_2, y_2)$$

Substituting (2.12)

$$f\big((x_1, y_1) \vee (x_2, y_2)\big) + f\big((x_1, y_1) \wedge (x_2, y_2)\big) \geq f(x_1, y_1) + f(x_2, y_2)$$

which establishes the supermodularity of f on $X \times Y$ (2.17).

2.60 In the standard Bertrand model of oligopoly

- the strategy space of each firm is $\Re_+$, a lattice.
- $u_i(p_i, \mathbf{p}_{-i})$ is supermodular in p_i (Exercise 2.51).
- If the other firm's increase their prices from $\mathbf{p}^1_{-i}$ to $\mathbf{p}^2_{-i}$, the effect on the demand for firm i's product is

$$f(p_i, \mathbf{p}^2_{-i}) - f(p_i, \mathbf{p}^1_{-i}) = \sum_{i \neq j} d_{ij}(p^2_j - p^1_j)$$

  If the goods are gross substitutes, demand for firm i increases and the amount of the increase is independent of p_i. Consequently, the effect on profit will be increasing in p_i. That is the payoff function (net revenue) has increasing differences in $(p_i, \mathbf{p}_{-i})$. Specifically,

$$u(p_i, \mathbf{p}^2_{-i}) - u(p_i, \mathbf{p}^1_{-i}) = \sum_{i \neq j} d_{ij}(p_i - \bar{c}_i)(p^2_j - p^1_j)$$

  For any price increase $\mathbf{p}^2_{-i} \gneqq \mathbf{p}^1_{-i}$, the change in profit $u(p_i, \mathbf{p}^2_{-i}) - u(p_i, \mathbf{p}^1_{-i})$ is increasing in p_i.

Hence, the Bertrand oligopoly model is a supermodular game.

2.61 Suppose f displays increasing differences so that for all $x_2 \succsim x_1$ and $y_2 \succsim y_1$

$$f(x_2, y_2) - f(x_1, y_2) \geq f(x_2, y_1) - f(x_1, y_1)$$

Then

$$f(x_2, y_1) - f(x_1, y_1) \geq 0 \implies f(x_2, y_2) - f(x_1, y_2) \geq 0$$

and

$$f(x_2, y_1) - f(x_1, y_1) > 0 \implies f(x_2, y_2) - f(x_1, y_2) > 0$$

2.62 For any $\boldsymbol{\theta} \in \Theta^*$, let $\mathbf{x}_1, \mathbf{x}_2 \in \varphi(\boldsymbol{\theta})$. Supermodularity implies

$$f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}) + f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}) \geq f(\mathbf{x}_1, \boldsymbol{\theta}) + f(\mathbf{x}_2, \boldsymbol{\theta})$$

which can be rearranged to give

$$f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}) - f(\mathbf{x}_2, \boldsymbol{\theta}) \geq f(\mathbf{x}_1, \boldsymbol{\theta}) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}) \tag{2.13}$$

However $\mathbf{x}_1$ and $\mathbf{x}_2$ are both maximal in $G(\boldsymbol{\theta})$.

$$\begin{aligned} f(\mathbf{x}_2, \boldsymbol{\theta}) \geq f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}) &\implies f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}) - f(\mathbf{x}_2, \boldsymbol{\theta}) \leq 0 \\ f(\mathbf{x}_1, \boldsymbol{\theta}) \geq f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}) &\implies f(\mathbf{x}_1, \boldsymbol{\theta}) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}) \geq 0 \end{aligned}$$

Substituting in (2.13), we conclude

$$0 \geq f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}) - f(\mathbf{x}_2, \boldsymbol{\theta}) \geq f(\mathbf{x}_1, \boldsymbol{\theta}) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}) \geq 0$$

This inequality must be satisfied as an equality with

$$\begin{aligned} f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}) &= f(\mathbf{x}_2, \boldsymbol{\theta}) \\ f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}) &= f(\mathbf{x}_1, \boldsymbol{\theta}) \end{aligned}$$

That is $\mathbf{x}_1 \vee \mathbf{x}_2 \in \varphi(\boldsymbol{\theta})$ and $\mathbf{x}_1 \wedge \mathbf{x}_2 \in \varphi(\boldsymbol{\theta})$. By Exercise 2.45, φ has an increasing selection.

2.63 As in the proof of the theorem, let $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ belong to Θ with $\boldsymbol{\theta}_2 \succsim \boldsymbol{\theta}_1$. Choose any optimal solutions $\mathbf{x}_1 \in \varphi(\boldsymbol{\theta}_1)$ and $\mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_2)$. We claim that $\mathbf{x}_2 \succsim_X \mathbf{x}_1$. Assume otherwise, that is assume $\mathbf{x}_2 \not\succsim_X \mathbf{x}_1$. This implies (Exercise 1.44) that $\mathbf{x}_1 \wedge \mathbf{x}_2 \neq \mathbf{x}_1$. Since $\mathbf{x}_1 \succsim \mathbf{x}_1 \wedge \mathbf{x}_2$, we must have $\mathbf{x}_1 \succ \mathbf{x}_1 \wedge \mathbf{x}_2$. Strictly increasing differences implies

$$f(\mathbf{x}_1, \boldsymbol{\theta}_2) - f(\mathbf{x}_1, \boldsymbol{\theta}_1) > f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_2) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_1)$$

which can be rearranged to give

$$f(\mathbf{x}_1, \boldsymbol{\theta}_2) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_2) > f(\mathbf{x}_1, \boldsymbol{\theta}_1) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_1) \tag{2.14}$$

Supermodularity implies

$$f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2) + f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_2) \geq f(\mathbf{x}_1, \boldsymbol{\theta}_2) + f(\mathbf{x}_2, \boldsymbol{\theta}_2)$$

which can be rearranged to give

$$f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2) - f(\mathbf{x}_2, \boldsymbol{\theta}_2) \geq f(\mathbf{x}_1, \boldsymbol{\theta}_2) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_2)$$

Combining this inequality with (2.14) gives

$$f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2) - f(\mathbf{x}_2, \boldsymbol{\theta}_2) > f(\mathbf{x}_1, \boldsymbol{\theta}_1) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_1) \tag{2.15}$$

However $\mathbf{x}_1$ and $\mathbf{x}_2$ are optimal for their respective parameter values, that is

$$\begin{aligned} f(\mathbf{x}_2, \boldsymbol{\theta}_2) \geq f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2) &\Longrightarrow f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2) - f(\mathbf{x}_2, \boldsymbol{\theta}_2) \leq 0 \\ f(\mathbf{x}_1, \boldsymbol{\theta}_1) \geq f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_1) &\Longrightarrow f(\mathbf{x}_1, \boldsymbol{\theta}_1) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_1) \geq 0 \end{aligned}$$

Substituting in (2.15), we conclude

$$0 \geq f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2) - f(\mathbf{x}_2, \boldsymbol{\theta}_2) > f(\mathbf{x}_1, \boldsymbol{\theta}_1) - f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_1) \geq 0$$

This contradiction implies that our assumption that $\mathbf{x}_2 \not\succsim_X \mathbf{x}_1$ is false. $\mathbf{x}_2 \succsim_X \mathbf{x}_1$ as required. φ is always increasing.

2.64 The budget correspondence is descending in $\mathbf{p}$ and therefore ascending in $-\mathbf{p}$. Consequently, the indirect utility function

$$v(\mathbf{p}, m) = \sup_{\mathbf{x} \in X(\mathbf{p}, m)} u(\mathbf{x})$$

is increasing in $-\mathbf{p}$, that is decreasing in $\mathbf{p}$.

2.65 $\boxed{\Longleftarrow}$ Let $\boldsymbol{\theta}_2 \succsim \boldsymbol{\theta}_1$ and $G_2 \succsim_S G_1$. Select $\mathbf{x}_1 \in \varphi(\boldsymbol{\theta}_1, G_1)$ and $\mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_2, G_2)$. Since $G_2 \succsim_S G_1$, $\mathbf{x}_1 \wedge \mathbf{x}_2 \in G_1$. Since $\mathbf{x}_1$ is optimal ($\mathbf{x}_1 \in \varphi(\boldsymbol{\theta}_1, G_1)$), $f(\mathbf{x}_1, \boldsymbol{\theta}_1) \geq f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_1)$. Quasisupermodularity implies $f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_1) \geq f(\mathbf{x}_2, \boldsymbol{\theta}_1)$. By the single crossing condition $f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2) \geq f(\mathbf{x}_2, \boldsymbol{\theta}_2)$. Therefore $\mathbf{x}_1 \vee \mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_2, G_2)$.

Similarly, since $G_2 \succsim_S G_1$, $\mathbf{x}_1 \vee \mathbf{x}_2 \in G(\boldsymbol{\theta}_2)$. But $\mathbf{x}_2$ is optimal, which implies that $f(\mathbf{x}_2, \boldsymbol{\theta}_2) \geq f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2)$ or $f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_2) \leq f(\mathbf{x}_2, \boldsymbol{\theta}_2)$. The single crossing condition implies that a similar inequality holds at $\boldsymbol{\theta}_1$, that is $f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}_1) \leq f(\mathbf{x}_2, \boldsymbol{\theta}_1)$. Quasisupermodularity implies that $f(\mathbf{x}_1, \boldsymbol{\theta}_1) \leq f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta}_1)$. Therefore $\mathbf{x}_1 \wedge \mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_1, G_1)$. Since $\mathbf{x}_1 \vee \mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_2, G_2)$ and $\mathbf{x}_1 \wedge \mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_1, G_1)$, φ is increasing in $(\boldsymbol{\theta}, G)$.

$\boxed{\Longrightarrow}$ To show that f is quasisupermodular, suppose that $\boldsymbol{\theta}$ is fixed. Choose any $\mathbf{x}_1, \mathbf{x}_2 \in X$. Let $G_1 = \{\mathbf{x}_1, \mathbf{x}_1 \wedge \mathbf{x}_2\}$ and $G_2 = \{\mathbf{x}_2, \mathbf{x}_1 \vee \mathbf{x}_2\}$. Then $G_2 \succsim_S G_1$. Assume that $f(\mathbf{x}_1, \boldsymbol{\theta}) \geq f(\mathbf{x}_1 \wedge \mathbf{x}_2, \boldsymbol{\theta})$. Then $\mathbf{x}_1 \in \varphi(\boldsymbol{\theta}, G_1)$ which implies that $\mathbf{x}_1 \vee \mathbf{x}_2 \in \varphi(\boldsymbol{\theta}, G_2)$. (If $\mathbf{x}_2 \in \varphi(\boldsymbol{\theta}, G_2)$, then also $\mathbf{x}_1 \vee \mathbf{x}_2 \in \varphi(\boldsymbol{\theta}, G_2)$ since φ is increasing in $(\boldsymbol{\theta}, G)$). But this implies that $f(\mathbf{x}_1 \vee \mathbf{x}_2, \boldsymbol{\theta}) \geq f(\mathbf{x}_2, \boldsymbol{\theta})$. f is quasisupermodular in X.

To show that f satisfies the single crossing condition, choose any $\mathbf{x}_2 \succsim \mathbf{x}_1$ and let $G = \{\mathbf{x}_1, \mathbf{x}_2\}$. Assume that $f(\mathbf{x}_2, \boldsymbol{\theta}_1) \geq f(\mathbf{x}_1, \boldsymbol{\theta}_1)$. Then $\mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_1, G)$ which implies that $\mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_2, G)$ for any $\boldsymbol{\theta}_2 \succsim \boldsymbol{\theta}_1$. (If $\mathbf{x}_1 \in \varphi(\boldsymbol{\theta}_2, G)$, then also $\mathbf{x}_1 \vee \mathbf{x}_2 = \mathbf{x}_2 \in \varphi(\boldsymbol{\theta}_2, G)$ since φ is increasing in $(\boldsymbol{\theta}, G)$.) But this implies that $f(\mathbf{x}_2, \boldsymbol{\theta}_2) \geq f(\mathbf{x}_1, \boldsymbol{\theta}_2)$. f satisfies the single crossing condition.

2.66 First, assume that f is continuous. Let T be an open subset in Y and $S = f^{-1}(T)$. If $S = \emptyset$, it is open. Otherwise, choose $x_0 \in S$ and let $y_0 = f(x_0) \in T$. Since T is open, there exists a neighborhood $N(y_0) \subseteq T$. Since f is continuous, there exists a corresponding neighborhood $N(x_0)$ with $f(N(x_0)) \subseteq N(f(x_0))$. Since $N(f(x_0)) \subseteq T$, $N(x_0) \subseteq S$. This establishes that for every $x_0 \in S$ there exist a neighborhood $N(x_0)$ contained in S. That is, S is open in X.

Conversely, assume that the inverse image of every open set in Y is open in X. Choose some $x_0 \in X$ and let $y_0 = f(x_0)$. Let $T \subset Y$ be a neighborhood of y_0. T contains an open ball $B_r(y_0)$ about y_0. By hypothesis, the inverse image $S = f^{-1}(B_r(y_0))$ is open in X. Therefore, there exists a neighborhood $N(x_0) \subseteq f^{-1}(B_r(y_0))$. Since $B_r(y_0) \subseteq T$, $f(N(x_0)) \subseteq T$. Since the choice of x_0 was arbitrary, we conclude that f is continuous.

2.67 Assume f is continuous. Let T be a closed set in Y and let $S = f^{-1}(T)$. Then, T^c is open. By the previous exercise, $f^{-1}(T^c) = S^c$ is open and therefore S is closed. Conversely, for every open set $T \subseteq Y$, T^c is closed. By hypothesis, $S^c = f^{-1}(T^c)$ is closed and therefore $S = f^{-1}(T)$ is open. f is continuous by the previous exercise.

2.68 Assume f is continuous. Let x^n be a sequence converging to x Let T be a neighborhood of $f(x)$. Since f is continuous, there exists a neighborhood $S \ni x$ such that $f(S) \subseteq T$. Since x^n converges to x, there exists some N such that $x^n \in S$ for all $n \geq N$. Consequently $f(x^n) \in T$ for every $n \geq N$. This establishes that $f(x^n) \to f(x)$.

Conversely, assume that for every sequence $x^n \to x$, $f(x^n) \to f(x)$. We show that if f were not continuous, it would be possible to construct a sequence which violates this hypothesis. Suppose then that f is not continuous. Then there exists a neighborhood T of $f(x)$ such that for every neighborhood S of x, there is $x' \in S$ with $f(x') \notin T$. In particular, consider the sequence of open balls $B_{1/n}(x)$. For every n, choose a point $x^n \in B_{1/n}(x)$ with $f(x^n) \notin T$. Then $x^n \to x$ but $f(x^n)$ does not converge to $f(x)$. This contradicts the assumption. We conclude that f must be continuous.

2.69 Since f is one-to-one and onto, it has an inverse $g = f^{-1}$ which maps Y onto X. Let S be an open set in X. Since f is open, $T = g^{-1}(S) = f(S)$ is open in Y. Therefore $g = f^{-1}$ is continuous.

2.70 Assume f is continuous. Let (x^n, y^n) be a sequence of points in graph(f) converging to (x, y). Then $y^n = f(x^n)$ and $x^n \to x$. Since f is continuous, $y = f(x) = \lim_{n\to\infty} f(x^n) = \lim_{n\to\infty} y^n$. Therefore $(x, y) \in \text{graph}(f)$ which is therefore closed.

2.71 By the previous exercise, f continuous implies graph(f) closed. Conversely, suppose graph(f) is closed and let x^n be a sequence converging to x. Then $(x^n, f(x^n)))$ is a sequence in graph(f). Since Y is compact, $f(x^n)$ contains a subsequence which converges y. Since graph(f) is closed, $(x, y) \in \text{graph}(f)$ and therefore $y = f(x)$ and $f(x^n) \to f(x)$.

2.72 Let T be an open set in Z. Since f and g are continuous, $g^{-1}(T)$ is open in Y and $f^{-1}(g^{-1}(T))$ is open in X. But $f^{-1}(g^{-1}(T)) = (f \circ g)^{-1}(T)$. Therefore $f \circ g$ is continuous.

2.73 Exercises 1.201 and 2.68.

2.74 Let u be defined as in Exercise 2.38. Let $(\mathbf{x}^n)$ be a sequence converging to $\mathbf{x}$. Let $z^n = u(\mathbf{x}^n)$ and $z = u(\mathbf{x})$. We need to show that $z^n \to z$.

(z^n) has a convergent subsequence. Let $\bar{z} = \max_i x_i$ and $\underline{z} = \min_i x_i$. Then $z \in [\underline{z}, \bar{z}]$. Fix some $\epsilon > 0$. Since $\mathbf{x}^n \to \mathbf{x}$, there exists some N such that $\|\mathbf{x}^n - \mathbf{x}\|_\infty < \epsilon$ for every $n \geq N$. Consequently, for all $n \geq N$, the terms of the sequence (z^n) lie in the compact set $[\underline{z} - \epsilon, \bar{z} + \epsilon]$. Hence, (z^n) has a convergent subsequence (z^m).

Every convergent subsequence (z^m) converges to z. Suppose not. That is, suppose there exists a convergent subsequence which converges to z'. Without loss of generality, assume $z' > z$. Let $\hat{z} = \frac{1}{2}(z + z')$ and let $\mathbf{z} = z\mathbf{1}$, $\mathbf{z}' = z'\mathbf{1}$, $\hat{\mathbf{z}} = \hat{z}\mathbf{1}$ be the corresponding commodity bundles (see Exercise 2.38). Since $z^m \to z' > \hat{z}$, there exists some M such that $z^m > \hat{z}$ for every $m \geq M$. This implies that

$$\mathbf{x}^m \sim \mathbf{z}^m \succ \hat{\mathbf{z}} \text{ for every } \ m \geq M$$

by monotonicity. Now $\mathbf{x}^m \to \mathbf{x}$ and continuity of preferences implies that $\mathbf{x} \succsim \hat{\mathbf{z}}$. However $\mathbf{x} \sim \mathbf{z}$ which implies that $\mathbf{z} \succsim \hat{\mathbf{z}}$ which contradicts monotonicity, since $\hat{\mathbf{z}} > \mathbf{z}$. Consequently, every convergent subsequence (z^m) converges to z.

2.75 Assume X is compact. Let y^n be a sequence in $f(X)$. There exists a sequence x^n in X with $y^n = f(x^n)$. Since X is compact, it contains a convergent subsequence $x^m \to x$. If f is continuous, the subsequence $y^m = f(x^m)$ converges in $f(X)$ (Exercise 2.68). Therefore $f(X)$ is compact.

Assume X is connected but $f(X)$ is not. This means there exists open subsets G and H in Y such that $f(X) \subset G \bigcup H$ and $(G \cap f(X)) \bigcap (H \cap f(X)) = \emptyset$. This implies that $X = f^{-1}(G) \cup f^{-1}(H)$ is a disconnection of X, which contradicts the connectedness of X.

2.76 Let S be any open set in X. Its complement S^c is closed and therefore compact. Consequently, $f(S^c)$ is compact (Exercise 2.3) and hence closed. Since f is one-to-one and onto, $f(S)$ is the complement of $f(S^c)$, and thus open in Y. Therefore, f is an open mapping. By Exercise 2.69, f^{-1} is continuous and f is a homeomorphism.

2.77 Assume f continuous. The sets $\{f(x) \geq a\}$ and $\{f(x) \leq a\}$ are closed subsets of the $\Re$ and hence $\succsim(a) = f^{-1}\{f(x) \geq a\}$ and $\precsim(a) = f^{-1}\{f(x) \leq a\}$ are closed subsets of X (Exercise 2.67).

Conversely, assume that all upper $\succsim(a)$ and lower $\precsim(a)$ contour sets are closed. This implies that the sets $\succ(a)$ and $\prec(a)$ are open.

Let A be an open set in $\Re$. Then for every $a \in A$, there exists an open ball $B_{r_a}(a) \subseteq A$

$$A = \bigcup_{a \in A} B_{r_a}(a)$$

For every $a \in A$, $B_{r_a}(a) = (a - r_a, a + r_a)$ and

$$f^{-1}\left(B_{r_a}(a)\right) = \succ(a - r_a) \cap \prec(a + r_a)$$

which is open. Consequently

$$f^{-1}(A) = \bigcup_{a \in A} f^{-1}\left(B_{r_a}(a)\right) = \bigcup_{a \in A} \left(\succ(a - r_a) \cap \prec(a + r_a)\right)$$

is open. f is continuous by Exercise 2.66.

2.78 Choose any $x_0 \in X$ and $\epsilon > 0$. Since f is continuous, there exists δ_1 such that

$$\rho(x, x_0) < \delta_1 \implies |f(x) - f(x_0)| < \epsilon/2$$

Similarly, there exists δ_2 such that

$$\rho(x, x_0) < \delta_2 \implies |g(x) - g(x_0)| < \epsilon/2$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, provided $\rho(x, x_0) < \delta$

$$\begin{aligned} |(f+g)(x) - (f+g)(x_0)| &= |f(x) + g(x) - f(x_0) - g(x_0)| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \epsilon \end{aligned}$$

This establishes $f + g$ is continuous at x_0. Since x_0 was arbitrary, $f + g$ is continuous for every $x_0 \in X$. The continuity of αf is shown similarly.

2.79 Choose any $x_0 \in X$. Given $0 < \eta \leq 1$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \eta \text{ and } \quad |g(x) - g(x_0)| < \eta$$

whenever $\rho(x, x_0) < \delta$. Consequently, while $\rho(x, x_0) < \delta$

$$\begin{aligned} |f(x)| &\leq |f(x) - f(x_0)| + |f(x_0)| \\ &< \eta + |f(x_0)| \\ &\leq 1 + |f(x_0)| \end{aligned}$$

and

$$\begin{aligned} |(fg)(x) - (fg)(x_0)| &= |f(x)g(x) - f(x_0)g(x_0)| \\ &= |f(x)(g(x) - g(x_0)) + g(x_0)(f(x) - f(x_0))| \\ &\leq |f(x)|\,|g(x) - g(x_0)| + |g(x_0)|\,|f(x) - f(x_0)| \\ &< \eta(1 + |f(x_0)| + |g(x_0)|) \end{aligned}$$

Given $\epsilon > 0$, let $\eta = \min\{1, \epsilon/(1 + |f(x_0)| + |g(x_0)|)\}$. Then, we have shown that there exists $\delta > 0$ such that

$$\rho(x, x_0) < \delta \implies |(fg)(x) - (fg)(x_0)| < \epsilon$$

Therefore, fg is continuous at x_0.

2.80 Apply Exercises 2.78 and 2.72.

2.81 For any $a \in \Re$, the upper and lower contour sets of $f \vee g$, namely

$$\{\, x : \max\{f(x), g(x)\} \geq a\} = \{x : f(x) \geq a\,\} \cup \{\, x : g(x) \geq a\,\}$$

$$\{\, x : \max\{f(x), g(x)\} \leq a\} = \{x : f(x) \leq a\,\} \cap \{\, x : g(x) \leq a\,\}$$

are closed. Therefore $f \vee g$ is continuous (Exercise 2.77). Similarly for $f \wedge g$.

2.82 The set $T = f(X)$ is compact (Proposition 2.3). We want to show that T has both largest and smallest elements. Assume otherwise, that is assume that T has no largest element. Then, the set of intervals $\{(-\infty, t) : t \in T\}$ forms an open covering of T. Since T is compact, there exists a finite subcollection of intervals $\{(-\infty, t_1), (-\infty, t_2), \ldots, (-\infty, t_n)\}$ which covers T. Let t^* be the largest of these t_i. Then t^* does not belong to any of the intervals $\{(-\infty, t_1), (-\infty, t_2), \ldots, (-\infty, t_n)\}$, contrary to the fact that they cover T. This contradiction shows that, contrary to our assumption, there must exist a largest element $t^* \in T$, that is $t^* \geq t$ for all $t \in T$. Let $x^* \in f^{-1}(t^*)$. Then $t^* = f(x^*) \geq f(x)$ for all $x \in X$. The existence of a smallest element is proved analogously.

2.83 By Proposition 2.3, $f(X)$ is connected and hence an interval (Exercise 1.95).

2.84 The range $f(X)$ is a compact subset of $\Re$ (Proposition 2.3). Therefore f is bounded (Proposition 1.1).

2.85 Let $\tilde{C}(X)$ denote the set of all continuous (not necessarily bounded) functionals on X. Then

$$C(X) = B(X) \cap \tilde{C}(X)$$

$B(X)$, $\tilde{C}(X)$ are a linear subspaces of the set of all functionals $F(X)$ (Exercises 2.11, 2.78 respectively). Therefore $C(X) = B(X) \cap \tilde{C}(X)$ is a subspace of $F(X)$ (Exercise 1.130). Clearly $C(X) \subseteq B(X)$. Therefore $C(X)$ is a linear subspace of $B(X)$.

Let f be a bounded function in the closure of $C(X)$, that is $f \in \overline{C(X)}$. We show that f is continuous. For any $\epsilon > 0$, there exists $f_0 \in C(X)$ such that $\|f - f_0\| < \epsilon/3$. Therefore $|f(x) - f_0(x)| < \epsilon/3$ for every $x \in X$. Choose some $x_0 \in X$. Since f_0 is continuous, there exists $\delta > 0$ such that

$$\rho(x, x_0) < \delta \implies |f_0(x) - f_0(x_0)| < \epsilon/3$$

Therefore, for every $x \in X$ such that $\rho(x, x_0) < \delta$

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_0(x) + f_0(x) - f_0(x_0) + f_0(x_0) - f(x_0)| \\ &\leq |f(x) - f_0(x)| + |f_0(x) - f_0(x_0)| + |f_0(x_0) - f(x_0)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

Therefore f is continuous at x_0. Since x_0 was arbitrary, we conclude that is continuous everywhere, that is $f \in C(X)$. Therefore $C(X) = \overline{C(X)}$ and $C(X)$ is closed in $B(X)$.

Since $B(X)$ is complete (Exercise 2.11), we conclude that $C(X)$ is complete (Exercise 1.107). Therefore $C(X)$ is a Banach space.

2.86 For every $\alpha \in \Re$,

$$\{\, x : f(x) \geq \alpha \,\} = \{x : -f(x) \leq -\alpha \,\}$$

and therefore

$$\{\, x : f(x) \geq \alpha \,\} \text{ is closed} \iff \{x : -f(x) \leq -\alpha \,\} \text{ is closed}$$

2.87 Exercise 2.77.

2.88 **1 implies 2** Suppose f is upper semi-continuous. Let x^n be a sequence converging to x_0. Assume $f(x^n) \to \mu$. For every $\alpha < \mu$, there exists some N such that $f(x^n) > \alpha$ for every $n \geq N$. Hence

$$x_0 \in \overline{\{\, x : f(x) \geq \alpha \,\}} = \{\, x : f(x) \geq \alpha \,\}$$

since f is upper semi-continuous. That is, $f(x_0) \geq \alpha$ for every $\alpha < \mu$. Hence $f(x_0) \geq \mu = \lim_{n\to\infty} f(x^n)$.

2 implies 3 Let (x^n, y^n) be a sequence in hypo f which converges to (x, y). That is, $x^n \to x$, $y^n \to y$ and $y^n \leq f(x^n)$. Condition 2 implies that $f(x) \geq y$. Hence, $(x, y) \in$ hypo f. Therefore hypo f is closed.

3 implies 1 For fixed $\alpha \in \Re$, let x^n be a sequence in $\{\, x : f(x) \geq \alpha \,\}$. Suppose $x^n \to x_0$. Then, the sequence (x^n, α) converges to $(x_0, \alpha) \in$ hypo f. Hence $f(x_0) \geq \alpha$ and $x_0 \in \{\, x : f(x) \geq \alpha \,\}$, which is therefore closed (Exercise 1.106).

2.89 Let $M = \sup_{x \in X} f(x)$, so that

$$f(x) \leq M \text{ for every } \; x \in X \tag{2.16}$$

There exists a sequence x^n in X with $f(x^n) \to M$. Since X is compact, there exists a convergent subsequence $x^m \to x^*$ and $f(x^m) \to M$. However, since f is upper semi-continuous, $f(x^*) \geq \lim f(x^m) = M$. Combined with (2.16), we conclude that $f(x^*) = M$.

2.90 Choose some $\epsilon > 0$. Since f is uniformly continuous, there exists some $\delta > 0$ such that $\rho(f(x^m), f(x^n)) < \epsilon$ for every $x^m, x^n \in X$ such that $\rho(x^m, x^n) < \delta$. Let (x^n) be a Cauchy sequence in X. There exists some N such that $\rho(x^m, x^n) < \delta$ for every $m, n \geq N$. Uniform continuity implies that $\rho(f(x^m), f(x^n)) < \epsilon$ for every $m, n \geq N$. $(f(x^n))$ is a Cauchy sequence.

2.91 Suppose not. That is, suppose f is continuous but not uniformly continuous. Then there exists some $\epsilon > 0$ such that for $n = 1, 2, \dots$, there exist points x_n^1, x_n^2 such that

$$\rho(x_n^1, x_n^2) < 1/n \text{ but } \rho(f(x_n^1), f(x_n^2)) \geq \epsilon \tag{2.17}$$

Since X is compact, (x_n^1) has a subsequence (x_m^1) converging to some $x \in X$. By construction $(\rho(x_n^1, x_n^2) < 1/n)$, the sequence (x_m^2) also converges to x and by continuity

$$\lim_{m \to \infty} f(x_m^1) = \lim_{m \to \infty} f(x_m^2)$$

which contradicts (2.17).

2.92 Assume f is Lipschitz with constant β. For any $\epsilon > 0$, let $\delta = \epsilon / 2\beta$. Then, provided $\rho(x, x_0) \leq \delta$

$$\rho(f(x), f(x_0)) \leq \beta \rho(x, x_0) = \beta \delta = \beta \frac{\epsilon}{2\beta} = \frac{\epsilon}{2} < \epsilon$$

f is uniformly continuous.

2.93 Let $f, g \in B(X)$.F Since $B(X)$ is a normed linear space, for every $x \in X$

$$f(x) - g(x) = (f - g)(x) \leq \|f - g\|$$

which implies that

$$f(x) \leq g(x) + \|f - g\|$$

Since T is increasing and satisfies (2.21)

$$T(f) \leq T(g + \|f - g\|) = T(g) + \beta \|f - g\|$$

or

$$T(f) - T(g) \leq \beta \|f - g\|$$

That is, for every $x \in X$

$$(Tf - Tg)(x) \leq \beta \|f - g\|$$

and consequently

$$\|Tf - Tg\| \leq \beta \|f - g\|$$

T is a contraction with modulus β.

2.94 We have previously shown that T is increasing (Exercise 2.42). By direct calculation, for any constant $c \in \Re$,

$$\begin{aligned} T(v+c)(x) &= \sup_{y \in G(x)} \Big\{ f(x,y) + \beta\big(v(y)+c\big) \Big\} \\ &= \sup_{y \in G(x)} \Big\{ f(x,y) + \beta v(y) \Big\} + \beta c \\ &= T(v)(x) + \beta c \end{aligned}$$

2.95 Assume that F is a compact subset of $C(X)$. Then F is bounded (Proposition 1.1). To show that F is equicontinuous, choose $\epsilon > 0$. F is totally bounded (Exercise 1.113), so that there exist finite set of functions $\{f_1, f_2, \dots, f_n\}$ in F such that

$$\min_{k=1}^{n} \|f - f_k\| \leq \epsilon/3$$

Each f_k is uniformly continuous (Exercise 2.91), so that there exists $\delta_k > 0$ such that

$$\rho(x, x_0) \leq \delta \implies \rho(f_k(x), f_k(x_0) < \epsilon/3$$

Let $\delta = \min\{\delta_1, \delta_2, \dots, \delta_k\}$. Given any $f \in F$, let k be such that $\|f - f_k\| < \epsilon/3$. Then for any $x, x_0 \in X$, $\rho(x, x_0) \leq \delta$ implies

$$\rho(f(x), f(x_0) \leq \rho(f(x), f_k(x)) + \rho(f_k(x), f_k(x_0)) + \rho(f_k(x_0), f(x_0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for every $f \in F$. Therefore, F is equicontinuous.

Conversely, assume that $F \subseteq C(X)$ is closed, bounded and equicontinuous. Let (f_n) be a bounded equicontinuous sequence of functions in F. We show that (f_n) has a convergent subsequence.

1. First, we show that for any $\epsilon > 0$, there is exists a subsequence (f_m) such that $\|f_m - f_{m'}\| < \epsilon$ for every f_m, $f_{m'}$ in the subsequence. Since the functions are equicontinuous, there exists $\delta > 0$ such that

   $$\rho(f_n(x) - f_n(x_0) < \frac{\epsilon}{3}$$

   for every x, x_0 in X with $\rho(x, x_0) \leq \delta$. Since X is compact, it is totally bounded (Exercise 1.113). That is, there exist a finite number of open balls $B_\delta(x_i)$, $i = 1, 2 \dots, k$ which cover X. The sequence $(f_n(x_1), f_n(x_2, \dots, f_n(x_k))$ is a bounded sequence in $\Re^n$. By the Bolzano-Weierstrass theorem (Exercise 1.119), this sequence has a convergent subsequence $(f_m(x_1), f_m(x_2), \dots, f_m(x_k))$ such that $f_m(x_i) - f_{m'}(x_i) < \epsilon/3$ for i and every $f_m, f_{m'}$ in the subsequence. Consequently, for any $x \in X$, there exists i such that

   $$\begin{aligned} \rho(f_m(x), f_{m'}(x) &\leq \rho(f_m(x), f_m(x_i)) + \rho(f_m(x_i), f_{m'}(x_i)) + \rho(f_{m'}(x_i), f_{m'}(x)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

   That is, $\|f_m - f_{m'}\| < \epsilon$ for every f_m, $f_{m'}$ in the subsequence.

2. Choose a ball B_1 of radius 1 in $C(X)$ which contains infinitely many elements of (f_n). Applying step 1, there exists a ball B_2 of radius 1/2 containing infinitely many elements of (f_n). Proceeding in this fashion, we obtain a nested sequence $B_1 \supseteq B_2 \supseteq \dots$ of balls in $C(X)$ such that (a) $d(B_i) \to 0$ and (b) each B_i contains infinitely many terms of (f_n). Choosing $f_{n_i} \in B_i$ gives a convergent subsequence.

2.96 Let $g \in \overline{F}$. Then for every $\epsilon > 0$ there exists $\delta > 0$ and $f \in F$ such that $\|f - g\| < \epsilon/3$ and

$$\rho(x, x_0) \le \delta \implies \rho(f(x), f(x_0) < \epsilon/3$$

so that if $\rho(x, x_0) \le \delta$

$$\|g(x) - g(x_0)\| \le \|f(x) - g(x)\| + \|f(x) - f(x_0)\| + \|f(x_0) - g(x_0)\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

2.97 For every $T \subseteq Y$

$$\varphi^{-}(T^c) = \{\, x \in X : \varphi(x) \cap T^c \ne \emptyset \,\}$$
$$\varphi^{+}(T) = \{\, x \in X : \varphi(x) \subseteq T \,\}$$

For every $\mathbf{x} \in X$ *either* $\varphi(x) \subseteq T$ *or* $\varphi(x) \cap T^c \ne \emptyset$ but not both. Therefore

$$\varphi^{+}(T) \cup \varphi^{-}(T^c) = X$$
$$\varphi^{+}(T) \cap \varphi^{-}(T^c) = \emptyset$$

That is

$$\varphi^{+}(T) = \left(\varphi^{-}(T^c)\right)^c$$

2.98 Assume $x \in \varphi(T)^{-1}$. Then $\varphi(x) = T$, $\varphi(x) \subseteq T$ and $x \in \varphi^{+}(T)$. Now assume $x \in \varphi^{+}(T)$ so that $\varphi(x) \subseteq T$. Consequently, $\varphi(x) \cap T = \varphi(x) \ne \emptyset$ and $x \in \varphi^{-}(T)$.

2.99 The respective inverses are:

	φ_2^{-1}	φ_2^{+}	φ_2^{-}
$\{t_1\}$	$\emptyset$	$\emptyset$	$\{s_1\}$
$\{t_2\}$	$\emptyset$	$\emptyset$	$\{s_1, s_2\}$
$\{t_1, t_2\}$	$\{s_1\}$	$\{s_1\}$	$\{s_1, s_2\}$
$\{t_2, t_3\}$	$\{s_2\}$	$\{s_2\}$	$\{s_1, s_2\}$
$\{t_1, t_2, t_3\}$	$\emptyset$	$\{s_1, s_2\}$	$\{s_1, s_2\}$

2.100 Let T be an open interval meeting $\varphi(1)$, that is $\varphi(1) \cap T \ne \emptyset$. Since $\varphi(1) = \{1\}$, we must have $1 \in T$ and therefore $\varphi(x) \cap T \ne \emptyset$ for every $x \in X$. Therefore φ is lhc at $x = 1$. On the other hand, the open interval $T = (1/2, 3/2)$ contains $\varphi(1)$ but it does not contain $\varphi(x)$ for any $x > 1$. Therefore, φ is not uhc at $x = 1$.

2.101 Choose any open set $T \subseteq Y$ and $x \in X$. Since $\varphi(x) = K = \varphi(x')$ for every $x, x' \in X$

- $\varphi(x) \subseteq T$ if and only if $\varphi(x') \subseteq T$ for every $x, x' \in X$
- $\varphi(x) \cap T \ne \emptyset$ if and only if $\varphi(x') \cap T \ne \emptyset$ for every $x, x' \in X$.

Consequently, φ is both uhc and lhc at all $x \in X$.

2.102 First assume that the φ is uhc. Let T be any open subset in Y and $S = \varphi^{+}(T)$. If $S = \emptyset$, it is open. Otherwise, choose $x_0 \in S$ so that $\varphi(x_0) \subseteq T$. Since φ is uhc, there exists a neighborhood $S(x_0)$ such that $\varphi(x) \subseteq T$ for every $x \in S(x_0)$. That is, $S(x_0) \subseteq \varphi^{+}(T) = S$. This establishes that for every $x_0 \in S$ there exist a neighborhood $S(x_0)$ contained in S. That is, S is open in X.

Conversely, assume that the upper inverse of every open set in Y is open in X. Choose some $x_0 \in X$ and let T be an open set containing $\varphi(x_0)$. Let $S = \varphi^{+}(T)$. S is an open set containing x_0. That is, S is a neighborhood of x_0 with $\varphi(x) \subseteq T$ for every $x \in S$. Since the choice of x_0 was arbitrary, we conclude that φ is uhc.

The lhc case is analogous.

2.103 Assume φ is uhc and T be any closed set in Y. By Exercise 2.97

$$\varphi^-(T) = \left[\varphi^+(T^c)\right]$$

T^c is open. By the previous exercise, $\varphi^+(T^c)$ is open which implies that $\varphi^-(T)$ is closed.

Conversely, assume $\varphi^-(T)$ is closed for every closed set T. Let T be an open subset of Y so that T^c is closed. Again by Exercise 2.97,

$$\varphi^+(T) = \left[\varphi^-(T^c)\right]$$

By assumption $\varphi^-(T^c)$ is closed and therefore $\varphi^+(T)$ is open. By the previous exercise, φ is uhc.

The lhc case is analogous.

2.104 Assume that φ is uhc at x_0. We first show that (y^n) is bounded and hence has a convergent subsequence. Since $\varphi(x_0)$ is compact, there exists a bounded open set T containing $\varphi(x_0)$. Since φ is uhc, there exists a neighborhood S of x_0 such that $\varphi(x) \subseteq T$ for $x \in S$. Since $x^n \to x_0$, there exists some N such that $x^n \in S$ for every $n \geq N$. Consequently, $\varphi(x^n) \subseteq T$ for every $n \geq N$ and therefore $y^n \in T$ for every $n \geq N$. The sequence y^n is bounded and hence has a convergent subsequence $y^m \to y_0$.

To complete the proof, we have to show that $y_0 \in \varphi(x_0)$. Assume not, assume that $y_0 \notin \varphi(x_0)$. Then, there exists an open set T containing $\varphi(x_0)$ such that $y_0 \notin \overline{T}$ (Exercise 1.93). Since φ is uhc, there exists N such that $\varphi(x^n) \subseteq T$ for every $n \geq N$. This implies that $y^m \in T$ for every $m \geq N$. Since $y^m \to y_0$, we conclude that $y_0 \in \overline{T}$, contradicting the specification of T.

Conversely, suppose that for every sequence $x^n \to x_0$, $y^n \in \varphi(x^n)$, there is a subsequence of $y^m \to y_0 \in \varphi(x_0)$. Suppose that φ is not uhc at x_0. That is, there exists an open set $T \supseteq \varphi(x_0)$ such that every neighborhood contains some x with $\varphi(x) \not\subseteq T$. From the sequence of neighborhoods $B_{1/n}(x_0)$, we can construct a sequence $x^n \to x$ and $y^n \in \varphi(x^n)$ but $y^n \notin T$. Such a sequence cannot have a subsequence which converges to $y^0 \in \varphi(x_)$, contradicting the hypothesis. We conclude that φ must be uhc at x_0.

2.105 Assume that φ is lhc. Let x^n be a sequence converging to x_0 and $y_0 \in \varphi(x_0)$. Consider the sequence of open balls $B_{1/m}(y_0), m = 1, 2, \dots$. Note that every $B_{1/m}(y_0)$ meets $\varphi(x_0)$. Since φ is lhc, there exists a sequence (S^m) of neighborhoods of x_0 such that $\varphi(x) \cap B_{1/m} \neq \emptyset$ for every $x \in S^m$. Since $x^n \to x$, for every m, there exists some N_m such that $x^n \in S_m$ for every $n \geq N_m$. Without loss of generality, we can assume that $N_1 < N_2 < N_3 \dots$. We can now construct the desired sequence y^n. For each $n = 1, 2, \dots$, choose y^n in the set $\varphi(x^n) \cap B_{1/\text{m}}$ where $N_m \leq n \leq N_{m+1}$ since

$$n \geq N_m \implies x^n \in S_m \implies \varphi(x^n) \cap B_{1/m} \neq \emptyset$$

Since $y^n \in B_{1/\text{m}}(y_0)$, the sequence (y^n) converges to y_0 and $n \to \infty$.

Conversely, assume that φ is not lhc at x_0, that is there exists an open set T with $T \cap \varphi(x_0) \neq \emptyset$ such that every neighborhood $S \ni x_0$ contains some x with $\varphi(x) \cap T = \emptyset$. Therefore, there exists a sequence $x^n \to x$ with $\varphi(x) \cap T = \emptyset$. Choose any $y_0 \in \varphi(x_0) \cap T$. By assumption, there exists a sequence $y^n \to y$ with $y^n \in \varphi(x^n)$. Since T is open and $y_0 \in T$, there exists some N such that $y^n \in T$ for all $n \geq N$, for which $\varphi(y^n) \cap T \neq \emptyset$. This contradiction establishes that φ is lhc at x_0.

2.106 1. Assume φ is closed. For any $x \in X$, let (y^n) be a sequence in $\varphi(x)$. Since φ is closed, $y^n \to y \in \varphi(x)$. Therefore $\varphi(x)$ is closed.

2. Assume φ is closed-valued and uhc. Choose any $(x, y) \notin \text{graph}(\varphi)$. Since $\varphi(x)$ is closed, there exist disjoint open sets T_1 and T_2 in Y such that $y \in T_1$ and $\varphi(x) \subseteq T_2$ (Exercise 1.93). Since φ is uhc, $\varphi^+(T_2)$ is a neighborhood of x. Therefore $\varphi^+(T_2) \times T_1$ is a neighborhood of (x, y) disjoint from $\text{graph}(\varphi)$. Therefore the complement of $\text{graph}(\varphi)$ is open, which implies that $\text{graph}(\varphi)$ is closed.

3. Since φ is closed and Y compact, φ is compact-valued. Let $(x^n) \to x$ be a sequence in X and (y^n) a sequence in Y with $y^n \in \varphi(x^n)$. Since Y is compact, there exists a subsequence $y^m \to y$. Since φ is closed, $y \in \varphi(x)$. Therefore, by Exercise 2.104, φ is uhc.

2.107 Assume φ is closed-valued and uhc. Then φ is closed (Exercise 2.106). Conversely, if φ is closed, then $\varphi(x)$ is closed for every x (Exercise 2.106). If Y is compact, then φ is compact-valued (Exercise 1.110). By Exercise 2.104, φ is uhc.

2.108 φ_1 is closed-valued (Exercise 2.106). Similarly, φ_2 is closed-valued (Proposition 1.1). Therefore, for every $x \in X$, $\varphi(x) = \varphi_1(x) \cap \varphi_2(x)$ is closed (Exercise 1.85) and hence compact (Exercise 1.110). Hence φ is compact-valued.

Now, for any $x_0 \in X$, let T be an open neighborhood of $\varphi(x_0)$. We need to show that there is a neighborhood S of x_0 such that $\varphi(S) \subseteq T$.

Case 1 $T \supseteq \varphi_2(x_0)$**:** Since φ_2 is uhc, there exists a neighborhood such that $S \ni x_0$ such that $\varphi_2(S) \subseteq T$ which implies that $\varphi(S) \subseteq \varphi_2(S) \subseteq T$

Case 2 $T \not\supseteq \varphi_2(x_0)$**:** Let $K = \varphi_2(x_0) \setminus T \neq \emptyset$. For every $y \in K$, there exist neighborhoods $S_y(x_0)$ and $T(y)$ such that $\varphi_1(S_y(x_0)) \cap T(y) = \emptyset$ (Exercise 1.93). The sets $T(y)$ constitute an open covering of K. Since K is compact, there exists a finite subcover, that is there exists a finite number of elements $y_1, y_2, \dots y_n$ such that

$$K \subseteq \bigcup_{i=1}^{n} T(y_i)$$

Let $T(K)$ denote $\bigcup_{i=1}^{n} T(y_i)$. Note that $T \cup T(K)$ is an open set containing $\varphi_2(x_0)$. Since φ_2 is uhc, there exists a neighborhood $S'(x_0)$ such that $\varphi_2(S'(x_0)) \subseteq T \cup T(K)$. Let

$$S(x_0) = \bigcap_{i=1}^{n} S_{y_i}(x_0) \cap S'(x_0)$$

$S(x_0)$ is an open neighborhood of x_0 for which

$$\varphi_1(S(x_0)) \cap T(K) = \emptyset \text{ and } \varphi_2(S(x_0)) \subseteq T \cup T(K)$$

from which we conclude that

$$\varphi(S(x_0)) = \varphi_1(S(x_0)) \cap \varphi_2(S(x_0)) \subseteq T$$

2.109 1. Let $\mathbf{x} \in X(\mathbf{p}, m) \cap T$. Then $\mathbf{x} \in X(\mathbf{p}, m)$ and $\sum_{i=1}^{n} p_i x_i \leq m$. Since T is open, there exists $\alpha < 1$ such that $\tilde{\mathbf{x}} = \alpha \mathbf{x} \in T$ and

$$\sum_{i=1}^{n} p_i \tilde{x}_i = \alpha \sum_{i=1}^{n} p_i x_i < \sum_{i=1}^{n} p_i x_i \leq m$$

2. (a) Suppose that $X(\mathbf{p}, m)$ is *not* lhc. Then for every neighborhood S of $(\mathbf{p}, m)$, there exists $(\mathbf{p}', m') \in S$ such that $X(\mathbf{p}', m') \cap T = \emptyset$. In particular, for every open ball $B_n(\mathbf{p}, m)$, there exists a point $(\mathbf{p}^n, m^n) \in B_n(\mathbf{p}, m)$ such that $X(\mathbf{p}^n, m^n) \cap T = \emptyset$. $((\mathbf{p}^n, m^n))$ is the required sequence.

   (b) By construction, $\|\mathbf{p}^n - \mathbf{p}\| < 1/n \to 0$ which implies that $p_i^n \to p_i$ for every i. Therefore (Exercise 1.202)

   $$\sum p_i^n \tilde{x}_i \to \sum p_i \tilde{x}_i < m \text{ and } m^n \to m$$

   and therefore there exists N such that

   $$\sum p_i^N \tilde{x}_i < m^N$$

   which implies that

   $$\tilde{\mathbf{x}} \in X(\mathbf{p}^N, m^N)$$

   (c) Also by construction $X(\mathbf{p}^N, m^N) \cap T = \emptyset$ which implies $X(\mathbf{p}^N, m^N) \subseteq T^c$ and therefore

   $$\tilde{\mathbf{x}} \in X(\mathbf{p}^n, m^n) \implies \tilde{\mathbf{x}} \notin T$$

   The assumption that $X(\mathbf{p}, m)$ is not lhc at $(\mathbf{p}, m)$ implies that $\tilde{\mathbf{x}} \notin T$, contradicting the conclusion in part 1 that $\tilde{\mathbf{x}} \in T$.

3. This contradiction establishes that $(\mathbf{p}, m)$ is lhc at $(\mathbf{p}, m)$. Since the choice of $(\mathbf{p}, m)$ was arbitrary, we conclude that the budget correspondence $X(\mathbf{p}, m)$ is lhc for all $(\mathbf{p}, m) \in P$ (assuming $X = \Re^n_+$).

4. In the previous example (Example 2.89), we have shown that $X(\mathbf{p}, m)$ is uhc. Hence, the budget correspondence is continuous for all $(\mathbf{p}, m)$ such that $m > \inf_{\mathbf{x} \in X} \sum_{i=1}^m p_i x_i$.

2.110 We give two alternative proofs.

Proof 1 Let $\mathcal{C} = \{S\}$ be an open cover of $\varphi(K)$. For every $x \in K$, $\varphi(x) \subseteq \varphi(K)$ is compact and hence can be covered by a finite number of the sets $S \in \mathcal{C}$. Let S_x denote the union of the finite cover of $\varphi(x)$. Since φ is uhc, every $\varphi^+(S_x)$ is open in X. Therefore $\{\, \varphi^+(S_x) : x \in K \,\}$ is an open covering of K. If K is compact, it contains an finite covering $\{\, \varphi^+(S_{x_1}), \varphi^+(S_{x_2}), \dots, \varphi^+(S_{x_n}) \,\}$. The sets $S_{x_1}, S_{x_2}, \dots, S_{x_n}$ are a finite subcovering of $\varphi(K)$.

Proof 2 Let (y^n) be a sequence in $\varphi(K)$. We have to show that (y^n) has a convergent subsequence with a limit in $\varphi(K)$. For every y^n, there is an x^n with $y^n \in \varphi(x^n)$. Since K is compact, the sequence (x^n) has a convergent subsequence $x^m \to x \in K$. Since φ is uhc, the sequence (y^m) has a subsequence (y^p) which converges to $y \in \varphi(x) \subseteq \varphi(K)$. Hence the original sequence (y^n) has a convergent subsequence.

2.111 The sets $X, \varphi(X), \varphi^2(X), \dots$ form a sequence of nonempty compact sets. Since $\varphi(X) \subseteq X$, $\varphi^2(X) \subseteq \varphi(X)$ and so on, the sequence of sets $\varphi^n X$ is decreasing. Let

$$K = \bigcap_{n=1}^{\infty} \varphi^n(X)$$

By the nested intersection theorem (Exercise 1.117), $K \neq \emptyset$. Since $K \subseteq \varphi^{n-1}(X)$, $\varphi(K) \subseteq \varphi^n(X)$ for every n, which implies that $\varphi(K) \subseteq K$.

To show that $K \subseteq \varphi(K)$, let $y \in K$. For every n there exists an $x^n \in \varphi^n(X)$ such that $y \in \varphi(x^n)$. Since X is compact, there exists a subsequence $x^m \to x_0$. Since $x^m \in \varphi^m(X)$ for every m, $x_0 \in K$. The sequence $(x^m, y) \to (x_0, y)$. Since φ is closed (Exercise 2.107), $y \in \varphi(x_0)$. Therefore $y \in \varphi(K)$ which implies that $K \subseteq \varphi(K)$.

2.112 $\varphi(x)$ is compact for every $x \in X$ by Tychonoff's theorem (Proposition 1.2). Let $x^k \to x$ be a sequence in X and let $y^k = (y_1^k, y_2^k, \dots, y_n^k)$ with $y_i^k \in \varphi(x^k)$ be a corresponding sequence of points in Y. For each y_i^k, $i = 1, 2, \dots, n$, there exists a subsequence $y_i^{k'} \to y_i$ with $y_i \in \varphi_i(x)$ (Exercise 2.104). Therefore $y = (y_1, y_2, \dots, y_n) \in \varphi(x)$ which implies that φ is uhc.

2.113 Let $v \in C(X)$. For every $\mathbf{x} \in X$, the maximand $f(x, y) + \beta v(y)$ is a continuous function on a compact set $G(x)$. Therefore the supremum is attained, and max can replace sup in the definition of the operator T (Theorem 2.2). Tv is the value function for the constrained optimization problem

$$\max_{y \in G(x)} \{ f(x, y) + \beta v(y) \}$$

satisfying the requirements of the continuous maximum theorem (Theorem 2.3), which ensures that Tv is continuous on X. We have previously shown that Tv is bounded (Exercise 2.18). Therefore $Tv \in C(X)$.

2.114

1. S has a least upper bound since X is a complete lattice. Let $s^* = \sup S$. Then $S^* = \succsim(s^*)$ is a complete sublattice of X (Exercise 1.48).
2. For every $s \in S$, $s \precsim s^*$ and since f is increasing and s is a fixed point

   $$s = f(s) \precsim f(s^*)$$

   Therefore $f(s^*) \in S^*$. ($f(s^*)$ is an upper bound of S). Again, since f is increasing, this implies that $f(x) \succsim f(s^*)$ for every $x \in S^*$. Therefore $f(S^*) \subseteq S^*$.
3. Let g be the restriction of f to the sublattice S^*. Since $f(S^*) \subseteq S^*$, g is an increasing function on a complete lattice. Applying Theorem 2.4, g has a smallest fixed point $\tilde{x}$.
4. $\tilde{x}$ is a fixed point of f, that is $\tilde{x} \in E$. Furthermore, $\tilde{x} \in S^*$. Therefore $\tilde{x}$ is an upper bound for S in E. Moreover, $\tilde{x}$ is the smallest fixed point of f in S^*. Therefore, $\tilde{x}$ is the least upper bound of S in E.
5. By Exercise 1.47, this implies that E is a complete lattice.

In Example 2.91, if $S = \{(2,1), (1,2)\}$, $S^* = \{(2,2), (3,2), (2,3), (3,3)\}$ and $\tilde{x} = (3,3)$.

2.115

1. For every $x \in M$, there exists some $y_x \in \varphi(x)$ such that $y_x \precsim x$. Moreover, since φ is increasing and $\tilde{x} \precsim x$, there exists some $z_x \in \varphi(\tilde{x})$ such that

   $$z_x \precsim y_x \precsim x \text{ for every } x \in M$$

2. Let $\tilde{z} = \inf\{z_x\}$
   (a) Since $z_x \precsim x$ for every $x \in M$, $\tilde{z} = \inf\{z_x\} \precsim \inf\{x\} = \tilde{x}$.
   (b) Since $\varphi(\tilde{x})$ is a complete sublattice of X, $\tilde{z} = \inf\{z_x\} \in \varphi(\tilde{x})$.
3. Therefore, $\tilde{x} \in M$.
4. Since $\tilde{z} \precsim \tilde{x}$ and φ is increasing, there exists some $y \in \varphi(\tilde{z})$ such that

   $$y \precsim \tilde{z} \in \varphi(\tilde{x})$$

   Hence $\tilde{z} \in M$.

5. This implies that $\tilde{x} \precsim \tilde{z}$. Therefore

$$\tilde{x} = \tilde{z} \in \varphi(\tilde{x})$$

$\tilde{x}$ is a fixed point of φ.

6. Since $E \subseteq M$, $\tilde{x} = \inf M$ is the least fixed point of φ.

2.116 1. Let $S \subseteq E$ and $s^* = \sup S$. For every $x \in S$, $x \in \varphi(x)$. Since φ is increasing, there exists some $z_x \in \varphi(s^*)$ such that $z_x \succsim x$.

2. Let $z^* = \sup z_x$. Then
   (a) Since $z_x \succsim x$ for every $x \in S$, $z^* = \sup z_x \succsim \sup x = s^*$
   (b) $z^* \in \varphi(s^*)$ since $\varphi(s^*)$ is a complete sublattice.

3. Define

$$S^* = \{\, x \in X : x \succsim s \text{ for every } \; s \in S \,\}$$

S^* is the set of all upper bounds of S in X. Then S^* is a complete lattice, since

$$S^* = \succsim(s^*)$$

4. Let $\mu \colon S^* \rightrightarrows S^*$ be the correspondence

$$\mu(x) = \varphi(x) \cap \psi(x)$$

where $\psi \colon S^* \rightrightarrows S^*$ is the constant correspondence defined by $\psi(x) = S^*$ for every $x \in S^*$. Then
   (a) Since φ is increasing, for every $x \succsim s^*$, there exists some $y_x \in \varphi(x)$ such that $y_x \succsim s^*$. Therefore $\mu(x) \neq \emptyset$ for every $x \in S^*$.
   (b) Both $\varphi(x)$ and $\psi(x)$ are complete sublattices for every $x \in S^*$. Therefore $\mu(x)$ is a complete sublattice for every $x \in S^*$.
   (c) Since both φ and ψ are increasing on S^*, μ is increasing on S^* (Exercise 2.47).

5. By the previous exercise, μ has a least fixed point $\tilde{x}$.

6. $\tilde{x} \in S^*$ is an upper bound of S. Therefore $\tilde{x}$ is the least upper bound of S in E.

7. By the previous exercise, E has a least element. Since we have shown every subset $S \subseteq E$ has a least upper bound, this establishes that E is complete lattice (Exercise 1.47).

2.117 For any i, let $\mathbf{a}^1_{-i}, \mathbf{a}^2_{-i} \in A_{-i}$ with $\mathbf{a}^2_{-i} \succsim \mathbf{a}^1_{-i}$. Let $\bar{a}^1_i = f(\mathbf{a}^1_{-i})$ and $\bar{a}^2_i = f(\mathbf{a}^2_{-i})$. We want to show that $\bar{a}^2_i \succsim \bar{a}^1_i$. Since $\bar{a}^1_i \in B(\mathbf{a}^1_{-i})$ and $B(\mathbf{a}_{-i})$ is increasing, there exists some $a_i \in B(\mathbf{a}^2_{-i})$ such that $a_i \succsim \bar{a}^1_i$. (Exercise 2.44). Therefore

$$\sup B(\mathbf{a}_{-i}) = \bar{a}^2_i \succsim a_i \succsim \bar{a}^1_i$$

$\bar{f}_i$ is increasing.

2.118 For any player i, their best response correspondence $B_i(\mathbf{a}_{-i})$ is

1. increasing by the monotone maximum theorem (Theorem 2.1).
2. a complete sublattice of A_i for every $\mathbf{a}_{-i} \in A_{-i}$ (Corollary 2.1.1).

The joint best response correspondence

$$B(\mathbf{a}) = B_1(\mathbf{a}_{-1}) \times B_2(\mathbf{a}_{-2}) \times \cdots \times B_n(\mathbf{a}_{-n})$$

is also

1. increasing (Exercise 2.46)
2. a complete sublattice of A for every $\mathbf{a} \in A$

Therefore, the best response correspondence $B(\mathbf{a})$ satisfies the conditions of Zhou's theorem, which implies that the set E of fixed points of B is a nonempty complete lattice. E is precisely the set of Nash equilibria of the game.

2.119 In proving the theorem, we showed that

$$\rho(x^n, x^{n+m}) \leq \frac{\beta^n}{1-\beta}\rho(x^0, x^1)$$

for every $m, n \geq 0$. Letting $m \to \infty$, $x^{n+m} \to x$ and therefore

$$\rho(x^n, x) \leq \frac{\beta^n}{1-\beta}\rho(x^0, x^1)$$

Similarly, for every $n, m \geq 0$

$$\begin{aligned}
\rho(x^n, x^{n+m}) &\leq \rho(x^n, x^{n+1}) + \rho(x^{n+1}, x^{n+2}) + \cdots + \rho(x^{n+m-1}, x^{n+m}) \\
&\leq (\beta + \beta^2 + \cdots + \beta^m)\rho(x^{n-1}, x^n) \\
&\leq \frac{\beta(1-\beta^m)}{1-\beta}\rho(x^{n-1}, x^n)
\end{aligned}$$

Letting $m \to \infty$, $x^{n+m} \to x$ and $\beta^m \to 0$ so that

$$\rho(x^n, x) \leq \frac{\beta}{1-\beta}\rho(x^{n-1}, x^n)$$

2.120 First observe that $f(x) \geq 1$ for every $x \geq 1$. Therefore $f\colon X \to X$. For any $x, z \in X$

$$\frac{f(x) - f(y)}{x - y} = \frac{x - y + \frac{2}{x} - \frac{2}{y}}{2(x-y)} = \frac{1}{2} - \frac{1}{xy}$$

Since $\frac{1}{xy} \leq 1$ for all $x, y \in X$

$$-\frac{1}{2} \leq \frac{f(x) - f(y)}{x - y} \leq \frac{1}{2}$$

so that

$$\left|\frac{f(x) - f(y)}{x - y}\right| = \frac{|f(x) - f(y)|}{|x - y|} \leq \frac{1}{2}$$

or

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y|$$

f is a contraction on X with modulus $1/2$.

X is closed and hence complete (Exercise 1.107). Therefore, f has a fixed point. That is, there exists $x_0 \in X$ such that

$$x_0 = f(x_0) = \frac{1}{2}(x_0 + \frac{2}{x_0})$$

Rearranging

$$2x_0^2 = x_0^2 + 2 \implies x_0^2 = 2$$

so that $x_0 = \sqrt{2}$.

Letting $x^0 = 2$

$$x^1 = \frac{1}{2}(2+1) = \frac{3}{2}$$

Using the error bounds in Corollary 2.5.1,

$$\begin{aligned}
\rho(x^n, \sqrt{2}) &\leq \frac{\beta^n}{1-\beta}\rho(x^0, x^1) \\
&= \frac{(1/2)^n}{1/2}1/2 \\
&= \frac{1}{2^n} \\
&= \frac{1}{1024} < 0.001
\end{aligned}$$

when $n = 10$. Therefore, we conclude that 10 iterations are ample to reduce the error below 0.001. Actually, with experience, we can refine this *a priori* estimate. In Example 1.64, we calculated the first five terms of the sequence to be

$$(2, 1.5, 1.416666666666667, 1.41421568627451, 1.41421356237469)$$

We observe that

$$\rho(x^3, x^4) = 1.41421568627451 - 1.41421356237469) = 0.0000212389982$$

so that using the second inequality of Corollary 2.5.1

$$\rho(x^4, \sqrt{2}) \leq \frac{1/2}{1/2}0.0000212389982 < 0.001$$

$x^4 = 1.41421356237469$ is the desired approximation.

2.121 Choose any $x^0 \in S$. Define the sequence $x^n = f(x^n) = f^n(x^0)$. Then (x^n) is a Cauchy sequence in S converging to x. Since S is closed, $x \in S$.

2.122 By the Banach fixed point theorem, f^N has a unique fixed point x. Let β be the Lipschitz constant of f^N. We have to show

x **is a fixed point of** f

$$\rho(f(x), x) = \rho(f(f^N(x), f^N(x)) = \rho(f^N(f(x), f^N(x)) \leq \beta\rho(f(x), x)$$

Since $\beta < 1$, this implies that $\rho(f(x), x) = 0$ or $f(x) = x$.

x **is the only fixed point of** f Suppose $z = f(z)$ is another fixed point of f. Then z is a fixed point of f^N and

$$\rho(x, z) = \rho(f^N(x), f^N(z)) \leq \beta\rho(x, z)$$

which implies that $x = z$.

2.123 By the Banach fixed point theorem, for every $\theta \in \Theta$, there exists $x_\theta \in X$ such that $f_\theta(x_\theta) = x_\theta$. Choose any $\theta_0 \in \Theta$.

$$\begin{aligned}
\rho(x_\theta, x_{\theta_0}) &= \rho(f_\theta(x_\theta), f_{\theta_0}(x_{\theta_0})) \\
&\leq \rho(f_\theta(x_\theta), f_\theta(x_{\theta_0})) + \rho(f_\theta(x_{\theta_0}), f_{\theta_0}(x_{\theta_0})) \\
&\leq \beta\rho(x_\theta, x_{\theta_0}) + \rho(f_\theta(x_{\theta_0}), f_{\theta_0}(x_{\theta_0})) \\
(1-\beta)\rho(x_\theta, x_{\theta_0}) &\leq \rho(f_\theta(x_{\theta_0}), f_{\theta_0}(x_{\theta_0})) \\
\rho(x_\theta, x_{\theta_0}) &\leq \frac{\rho(f_\theta(x_{\theta_0}), f_{\theta_0}(x_{\theta_0}))}{(1-\beta)} \to 0
\end{aligned}$$

as $\theta \to \theta_0$. Therefore $x_\theta \to x_{\theta_0}$.

2.124

1. Let $\mathbf{x}$ be a fixed point of f. Then $\mathbf{x}$ satisfies

$$\mathbf{x} = (I - A)\mathbf{x} + \mathbf{c} = \mathbf{x} - A\mathbf{x} + c$$

which implies that $A\mathbf{x} = c$.

2. For any $\mathbf{x}^1, \mathbf{x}^2 \in X$

$$\begin{aligned}
\left\| f(\mathbf{x}^1) - f(\mathbf{x}^2) \right\| &= \left\| (I - A)(\mathbf{x}^1 - \mathbf{x}^2) \right\| \\
&\leq \| I - A \| \left\| \mathbf{x}^1 - \mathbf{x}^2 \right\|
\end{aligned}$$

Since $a_{ii} = 1$, the norm of $I - A$ is

$$\| I - A \| = \max_i \sum_{j \neq i} |a_{ij}| = k$$

and

$$\left\| f(\mathbf{x}^1) - f(\mathbf{x}^2) \right\| \leq k \left\| \mathbf{x}^1 - \mathbf{x}^2 \right\|$$

By the assumption of strict diagonal dominance, $k < 1$. Therefore f is a contraction and has a unique fixed point $\mathbf{x}$.

2.125

1.

$$\begin{aligned}
\varphi(x) &= \{\, y^* \in G(x) : f(x, y^*) + \beta v(y^*) = v(x) \,\} \\
&= \{ y^* \in G(x) : f(x, y^*) + \beta v(y^*) = \sup_{y \in G(x)} \{ f(x, y) + \beta v(y) \} \} \\
&= \{ y^* \in G(x) : f(x, y^*) + \beta v(y^*) \geq f(x, y) + \beta v(y) \text{ for every } \ y \in G(x) \} \\
&= \arg \max_{y \in G(x)} \{ f(x, y) + \beta v(y) \}
\end{aligned}$$

2. $\varphi(x)$ is the solution correspondence of a standard constrained maximization problem, with x as parameter and y the decision variable. By assumption the maximand $f(x, y) = f(x, y) + \beta v(y)$ is continuous and the constraint correspondence $G(x)$ is continuous and compact-valued. Applying the continuous maximum theorem (Theorem 2.3), φ is nonempty, compact-valued and uhc.

3. We have just shown that $\varphi(x)$ is nonempty for every $x \in X$. Starting at x_0, choose some $x_1^* \in \varphi(x_0)$. Then choose $x_2^* \in \varphi(x_1^*)$. Proceeding in this way, we can construct a plan $\mathbf{x}^* = x_0, x_1^*, x_2^*, \ldots$ such that $x_{t+1}^* \in \varphi(x_t^*)$ for every $t = 0, 1, 2, \ldots$.

4. Since $x^*_{t+1} \in \varphi(x^*_t)$ for every t, $\mathbf{x}$ satisfies Bellman's equation, that is

$$v(x^*_t) = f(x^*_t, x^*_{t+1}) + \beta v(x^*_{t+1}), \qquad t = 0, 1, 2, \dots$$

Therefore $\mathbf{x}$ is optimal (Exercise 2.17).

2.126 1. In the previous exercise (Exercise 2.125) we showed that the set φ of solutions to Bellman's equation (Exercise 2.17) is the solution correspondence of the constrained maximization problem

$$\varphi(x) = \arg \max_{y \in G(x)} \{ f(x, y) + \beta v(y) \}$$

This problem satisfies the requirements of the monotone maximum theorem (Theorem 2.1), since the objective function $f(x, y) + \beta v(y)$

- supermodular in y
- displays strictly increasing differences in (x, y) since for every $x^2 \geq x^1$

$$f(x^2, y) + \beta v(y) - f(x^1, y) + \beta v(y) = f(x^2, y) - f(x^1, y)$$

- $G(x)$ is increasing.

By Corollary 2.1.2, $\varphi(x)$ is always increasing.

2. Let $\mathbf{x}^* = (x_0, x^*_1, x^*_2, \dots)$ be an optimal plan. Then (Exercise 2.17)

$$x^*_{t+1} \in \varphi(x^*_t), \qquad t = 0, 1, 2, \dots$$

Since φ is always increasing

$$x^*_t \geq x^*_{t-1} \implies x^*_{t+1} \geq x^*_t$$

for every $t = 1, 2, \dots$. Similarly

$$x^*_t \leq x^*_{t-1} \implies x^*_{t+1} \leq x^*_t$$

$\mathbf{x}^* = (x_0, x^*_1, x^*_2, \dots)$ is a monotone sequence.

2.127 Let $g(x) = f(x) - x$. g is continuous (Exercise 2.78) with

$$g(0) \geq 0 \text{ and } g(1) \leq 0$$

By the intermediate value theorem (Exercise 2.83), there exists some point $x \in [0, 1]$ with $g(x) = 0$ which implies that $f(x) = x$.

2.128 1. To show that a label $\min\{ i : \beta_i \leq \alpha_i \neq 0 \}$ exists for every $\mathbf{x} \in S$, assume to the contrary that, for some $\mathbf{x} \in S$, $\beta_i > \alpha_i$ for every $i = 0, 1, \dots, n$. This implies

$$\sum_{i=0}^{n} \beta_i > \sum_{i=0}^{n} \alpha_i = 1$$

contradicting the requirement that

$$\sum_{i=0}^{n} \beta_i = 1 \text{ for every } f(\mathbf{x}) \in S$$

2. The barycentric coordinates of vertex $\mathbf{x}_i$ are $\alpha_i = 1$ with $\alpha_j = 0$ for every $j \neq i$. Therefore the rule assigns vertex $\mathbf{x}_i$ the label i.
3. Similarly, if $\mathbf{x}$ belongs to a proper face of S, it coordinates relative to the vertices not in that face are 0, and it cannot be assigned a label corresponding to a vertex not in the face. To be concrete, suppose that $\mathbf{x} \in \operatorname{conv}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4\}$. Then

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \alpha_4 \mathbf{x}_4, \qquad \alpha_1 + \alpha_2 + \alpha_4 = 1$$

and $\alpha_i = 0$ for $i \notin \{1, 2, 4\}$. Therefore

$$\mathbf{x} \longmapsto \min\{\, i : \beta_i \leq \alpha_i \neq 0 \,\} \in \{1, 2, 4\}$$

2.129 1. Since S is compact, it is bounded (Proposition 1.1) and therefore it is contained in a sufficiently large simplex T.

2. By Exercise 3.74, there exists a continuous retraction $r \colon T \to S$. The composition $f \circ r \colon T \to S \subseteq T$. Furthermore as the composition of continuous functions, $f \circ r$ is continuous (Exercise 2.72). Therefore $f \circ r$ has a fixed point $\mathbf{x}^* \in T$, that is $f \circ r(\mathbf{x}^*) = \mathbf{x}^*$.
3. Since $f \circ r(\mathbf{x}) \in S$ for every $\mathbf{x} \in T$, we must have $f \circ r(\mathbf{x}^*) = \mathbf{x}^* \in S$. Therefore, $r(\mathbf{x}^*) = \mathbf{x}^*$ which implies that $f(\mathbf{x}^*) = \mathbf{x}^*$. That is, $\mathbf{x}^*$ is a fixed point of f.

2.130 Convexity of S is required to ensure that there is a continuous retraction of the simplex onto S.

2.131 1. $f(x) = x^2$ on $S = (0, 1)$ or $f(x) = x + 1$ on $S = \Re_+$.

2. $f(x) = 1 - x$ on $S = [0, 1/3] \cup [2/3, 1]$.
3. Let $S = [0, 1]$ and define

$$f(x) = \begin{cases} 1 & 0 \leq x < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

2.132 Suppose such a function exists. Define $f(\mathbf{x}) = -r(\mathbf{x})$. Then $f \colon B \to B$ continously, and has no fixed point since for

- $\mathbf{x} \in S$, $f(\mathbf{x}) = -r(\mathbf{x}) = -\mathbf{x} \neq \mathbf{x}$
- $\mathbf{x} \in B \setminus S$, $f(\mathbf{x}) \notin B \setminus S$ and therefore$f(\mathbf{x}) \neq \mathbf{x}$

Therefore f has no fixed point contradicting Brouwer's theorem.

2.133 Suppose to the contrary that f has no fixed point. For every $\mathbf{x} \in B$, let $r(\mathbf{z})$ denote the point where the line segment from $f(\mathbf{x})$ through $\mathbf{x}$ intersects the boundary S of B. Since f is continuous and $f(\mathbf{x}) \neq \mathbf{x}$, r is a continuous function from B to its boundary, that is a retraction, contradicting Exercise 2.132. We conclude that f must have a fixed point.

2.134 No-retraction $\Longrightarrow$ Brouwer Note first that the no-retraction theorem (Exercise 2.132) generalizes immediately to a closed ball about $\mathbf{0}$ of arbitrary radius. Assume that f is a continuous operator on a compact, convex set S in a finite dimensional normed linear space. There exists a closed ball B containing S (Proposition 1.1). Define $g \colon B \to S$ by

$$g(\mathbf{y}) = \{\, \mathbf{x} \in S : \mathbf{x} \text{ is closest to } \mathbf{y} \,\}$$

As in Exercise 2.129, g is well-defined, continuous and $g(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in S$. $f \circ g \colon B \to S \subseteq B$ and has a fixed point $\mathbf{x}^* = f(g(\mathbf{x}^*))$ by Exercise 2.133. Since

$f \circ g(\mathbf{x}) \in S$ for every $\mathbf{x} \in B$, we must have $f \circ g(\mathbf{x}^*) = \mathbf{x}^* \in S$. Therefore, $g(\mathbf{x}^*) = \mathbf{x}^*$ which implies that $f(\mathbf{x}^*) = \mathbf{x}^*$. That is, $\mathbf{x}^*$ is a fixed point of f.

Brouwer $\implies$ no-retraction Exercise 2.132.

2.135 Let Λ_k, $k = 1, 2, \dots$ be a sequence of simplicial partitions of S in which the maximum diameter of the subsimplices tend to zero as $k \to \infty$. By Sperner's lemma (Proposition 1.3), every partition Λ_k has a completely labeled subsimplex with vertices $\mathbf{x}_0^k, \mathbf{x}_1^k, \dots, \mathbf{x}_n^k$. By construction of an admissible labeling, each $\mathbf{x}_i^k$ belongs to a face containing $\mathbf{x}_i$, that is

$$\mathbf{x}_i^k \in \operatorname{conv}\{\mathbf{x}_i, \dots\}$$

and therefore

$$\mathbf{x}_i^k \in A_i, \qquad i = 0, 1, \dots, n$$

Since S is compact, each sequence $\mathbf{x}_i^k$ has a convergent subsequence $\mathbf{x}_i^{k'}$. Moreover, since the diameters of the subsimplices converge to zero, these subsequences must converge to the same point, say $\mathbf{x}^*$. That is,

$$\lim_{k' \to \infty} \mathbf{x}_i^{k'} = \mathbf{x}^*, \qquad i = 0, 1, \dots, n$$

Since the sets A_i are closed, $\mathbf{x}^* \in A_i$ for every i and therefore

$$\mathbf{x}^* \in \bigcap_{i=0}^{n} A_i \neq \emptyset$$

2.136 $\boxed{\implies}$ Let $f \colon S \to S$ be a continuous operator on an n-dimensional simplex S with vertices $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$. For $i = 0, 1, \dots, n$, let

$$A_i = \{\, \mathbf{x} \in S : \beta_i \le \alpha_i \,\}$$

where $\alpha_0, \alpha_1, \dots, \alpha_n$ and $\beta_0, \beta_1, \dots, \beta_n$ are the barycentric coordinates of $\mathbf{x}$ and $f(\mathbf{x})$ respectively. Then

- f continuous $\implies$ A_i closed for every $i = 0, 1, \dots, n$ (Exercise 1.106)
- Let $\mathbf{x} \in \operatorname{conv}\{\, \mathbf{x}_i : i \in I \,\}$ for some $I \subseteq \{\, 0, 1, \dots, n \,\}$. Then

$$\sum_{i \in I} \alpha_i = 1 = \sum_{i=0}^{n} \beta_i$$

  which implies that $\beta_i \le \alpha_i$ for some $i \in I$, so that $\mathbf{x} \in A_i$. Therefore

$$\operatorname{conv}\{\, \mathbf{x}_i : i \in I \,\} \subseteq \bigcup_{i \in I} A_i$$

Therefore the collection $A_0, A_1, \dots, A_n$ satisfies the hypotheses of the K-K-M theorem and their intersection is nonempty. That is, there exists

$$\mathbf{x}^* \in \bigcap_{i=0}^{n} A_i \neq \emptyset \text{ with } \beta_i^* \le \alpha_i^*, \qquad i = 0, 1, \dots, n$$

where α^* and β^* are the barycentric coordinates of $\mathbf{x}^*$ and $f(\mathbf{x}^*)$ respectively. Since $\sum \beta_i^* = \sum \alpha_i^* = 1$, this implies that

$$\beta_i^* = \alpha_i^* \qquad i = 0, 1, \dots, n$$

In other words, $f(\mathbf{x}^*) = \mathbf{x}^*$.

$\boxed{\Leftarrow}$ Let $A_0, A_1, \dots, A_n$ be closed subsets of an n dimensional simplex S with vertices $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n$ such that

$$\operatorname{conv}\{\, \mathbf{x}_i : i \in I \,\} \subseteq \bigcup_{i \in I} A_i$$

for every $I \subseteq \{\, 0, 1, \dots, n \,\}$. For $i = 0, 1, \dots, n$, let

$$g_i(\mathbf{x}) = \rho(\mathbf{x}, A_i)$$

For any $\mathbf{x} \in S$ with barycentric coordinates $\alpha_0, \alpha_1, \dots, \alpha_n$, define

$$f(\mathbf{x}) = \beta_0 \mathbf{x}_0 + \beta_1 \mathbf{x}_1 + \dots + \beta_n \mathbf{x}_n$$

where

$$\beta_i = \frac{\alpha_i + g_i(\mathbf{x})}{1 + \sum_{j=0}^n g_j(\mathbf{x})} \qquad i = 0, 1, \dots, n \tag{2.18}$$

By construction $\beta_i \geq 0$ and $\sum_{i=0}^n \beta_i = 1$. Therefore $f(\mathbf{x}) \in S$. That is, $f \colon S \to S$. Furthermore f is continuous. By Brouwer's theorem, there exists a fixed point x^* with $f(\mathbf{x}^*) = \mathbf{x}^*$. That is $\beta_i^* = \alpha_i^*$ for $i = 0, 1, \dots, n$.

Now, since the collection $A_0, A_1, \dots, A_n$ covers S, there exists some i for which $\rho(\mathbf{x}^*, A_i) = 0$. Substituting $\beta_i^* = \alpha_i^*$in (2.18) we have

$$\alpha_i^* = \frac{\alpha_i^*}{1 + \sum_{j=0}^n g_j(\mathbf{x}^*)}$$

which implies that $g_j(\mathbf{x}^*) = 0$ for every j. Since the A_i are closed, $\mathbf{x}^* \in A_i$ for every i and therefore

$$\mathbf{x}^* \in \bigcap_{i=0}^n A_i \neq \emptyset$$

2.137 To simplify the notation, let $z_k^+(\mathbf{p}) = \max\bigl(0, \mathbf{z}_i(\mathbf{p})\bigr)$. Assume $\mathbf{p}^*$ is a fixed point of g. Then for every $k = 1, 2, \dots, n$

$$p_k^* = \frac{p_k + z_k^+(\mathbf{p}^*)}{1 + \sum_{j=1}^n z_j^+(\mathbf{p}^*)}$$

Cross-multiplying

$$p_k^* + p_k^* \sum_{j=1}^n z_j^+(\mathbf{p}) = p_k^* + z_k^+(\mathbf{p}^*)$$

or

$$p_k^* \sum_{j=1}^n z_j^+(\mathbf{p}) = z_k^+(\mathbf{p}^*) \qquad k = 1, 2, \dots n$$

Multiplying each equation by $z_k(\mathbf{p})$ we get

$$p_k^* z_k(\mathbf{p}^*) \sum_{j=1}^n z_i^+(\mathbf{p}) = z_k(\mathbf{p}^*) z_k^+(\mathbf{p}^*) \qquad k = 1, 2, \dots n$$

Summing over k

$$\sum_{k=1}^{n} p_k^* z_k(\mathbf{p}^*) \sum_{j=1}^{n} z_i^+(\mathbf{p}) = \sum_{k=1}^{n} z_k(\mathbf{p}^*) z_k^+(\mathbf{p}^*)$$

Since $\sum_{k=1}^n p_k^* z_k(\mathbf{p}^*) = 0$ this implies that

$$\sum_{k=1}^{n} z_k(\mathbf{p}^*) z_k^+(\mathbf{p}^*) = 0$$

Each term of this sum is nonnegative, since it is either 0 or $\left(z_k(\mathbf{p}^*)\right)^2$. Consequently, every term must be zero which implies that $z_k(\mathbf{p}*) \leq 0$ for every $k = 1, 2, \dots, l$. In other words, $\mathbf{z}(\mathbf{p}^*) \leq \mathbf{0}$.

2.138 Every individual demand function $\mathbf{x}_i(\mathbf{p}, m)$ is continuous (Example 2.90) in $\mathbf{p}$ and m. For given endowment $\boldsymbol{\omega}_i$

$$m_i = \sum_{j=1}^{l} p_j \boldsymbol{\omega}_{ij}$$

is continuous in $\mathbf{p}$ (Exercise 2.78). Therefore the excess demand function

$$\mathbf{z}_i(\mathbf{p}) = \mathbf{x}_i(\mathbf{p}, m) - \boldsymbol{\omega}_i$$

is continuous for every consumer i and hence the aggregate excess demand function is continuous.

Similarly, the consumer's demand function $\mathbf{x}_i(\mathbf{p}, m)$ is homogeneous of degree 0 in $\mathbf{p}$ and m. For given endowment $\boldsymbol{\omega}_i$, the consumer's wealth is homogeneous of degree 1 in $\mathbf{p}$ and therefore the consumer's excess demand function $\mathbf{z}_i(\mathbf{p})$ is homogeneous of degree 0. So therefore is the aggregate excess demand function $\mathbf{z}(\mathbf{p})$.

2.139

$$\begin{aligned} \mathbf{z}(\mathbf{p}) &= \sum_{i=1}^{n} \mathbf{z}_i(\mathbf{p}) \\ &= \sum_{i=1}^{n} \left(\mathbf{x}_i(\mathbf{p}, m) - \boldsymbol{\omega}_i\right) \end{aligned}$$

and therefore

$$\mathbf{p}^T \mathbf{z}(\mathbf{p}) = \sum_{i=1}^{n} \mathbf{p}^T \mathbf{x}_i(\mathbf{p}, m) - \sum_{i=1}^{n} \mathbf{p}^T \boldsymbol{\omega}_i$$

Since preferences are nonsatiated and strictly convex, they are locally nonsatiated (Exercise 1.248) which implies (Exercise 1.235) that every consumer must satisfy his budget constraint

$$\mathbf{p}^T \mathbf{x}_i(\mathbf{p}, m) = \mathbf{p}^T \boldsymbol{\omega}_i \text{ for every } \; i = 1, 2, \dots, n$$

Therefore in aggregate

$$\mathbf{p}^T \mathbf{z}(\mathbf{p}) = \sum_{i=1}^{n} \mathbf{p}^T \mathbf{x}_i(\mathbf{p}, m) - \sum_{i=1}^{n} \mathbf{p}^T \boldsymbol{\omega}_i = \mathbf{0}$$

for every $\mathbf{p}$.

2.140 Assume there exists $\mathbf{p}^*$ such that $\mathbf{z}(\mathbf{p}^*) \leq \mathbf{0}$. That is

$$\mathbf{z}(\mathbf{p}^*) = \sum_{i=1}^{n} \mathbf{z}_i(\mathbf{p}) = \sum_{i=1}^{n} \big(\mathbf{x}_i(\mathbf{p}, m) - \boldsymbol{\omega}_i\big) = \sum_{i=1}^{n} \mathbf{x}_i(\mathbf{p}, m) - \sum_{i=1}^{n} \boldsymbol{\omega}_i \leq \mathbf{0}$$

or

$$\sum_{i \in N} \mathbf{x}_i \leq \sum_{i \in N} \boldsymbol{\omega}_i$$

Aggregate demand is less or equal to available supply.

Let $m_i^* = \sum_{j=1}^{l} p_j^* \boldsymbol{\omega}_{ij}$ denote the wealth of consumer i when the price system is $\mathbf{p}^*$ and let $\mathbf{x}_i^* = \mathbf{x}(\mathbf{p}^*, m^*)$ be his chosen consumption bundle. Then

$$\mathbf{x}_i^* \succsim \mathbf{x}_i \text{ for every } \ \mathbf{x}_i \in X(\mathbf{p}^*, m_i)$$

Let $\underline{\mathbf{x}}^* = (\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_n^*)$ be the allocation comprising these optimal bundles. The pair $(\mathbf{p}^*, \underline{\mathbf{x}}^*)$ is a competitive equilibrium.

2.141 For each $\mathbf{x}^k$, let S^k denote the subsimplex of Λ^k which contains $\mathbf{x}^k$ and let $\mathbf{x}_0^k, \mathbf{x}_1^k, \dots, \mathbf{x}_n^k$ denote the vertices of S^k. Let $\alpha_0^k, \alpha_1^k, \dots, \alpha_n^k$ denote the barycentric coordinates (Exercise 1.159) of $\mathbf{x}$ with respect to the vertices of S^k and let $\mathbf{y}_i^k = f^k(\mathbf{x}_i^k)$, $i = 0, 1, \dots, n$, denote the images of the vertices. Since S is compact, there exists subsequences $\mathbf{x}_i^{k'}$, $\mathbf{y}_i^{k'}$ and $\alpha^{k'}$ such that

$$\mathbf{x}_i^k \to \mathbf{x}_i^* \quad \mathbf{y}_i^k \to \mathbf{y}_i^* \text{ and } \ \alpha_i^k \to \alpha_i^* \qquad i = 0, 1, \dots, n$$

Furthermore, $\alpha_i^* \geq 0$ and $\alpha_0^* + \alpha_1^* + \dots + \alpha_n^* = 1$. Since the diameters of the subsimplices converge to zero, their vertices must converge to the same point. That is, we must have

$$\mathbf{x}_0^* = \mathbf{x}_1^* = \dots = \mathbf{x}_n^* = \mathbf{x}^*$$

By definition of f^k

$$f^k(\mathbf{x}^k) = \alpha_0^k f(\mathbf{x}_0^k) + \alpha_1^k f(\mathbf{x}_1^k) + \dots + \alpha_n^k f(\mathbf{x}_n^k)$$

Substituting $\mathbf{y}_i^k = f^k(\mathbf{x}_i^k)$, $i = 0, 1, \dots, n$ and recognizing that $\mathbf{x}^k$ is a fixed point of f^k, we have

$$x^k = f^k(\mathbf{x}^k) = \alpha_0^k \mathbf{y}_0^k + \alpha_1^k \mathbf{y}_1^k + \dots + \alpha_n^k \mathbf{y}_n^k$$

Taking limits

$$\mathbf{x}^* = \alpha_0^* \mathbf{y}_0^* + \alpha_1^* \mathbf{y}_1^* + \dots + \alpha_n^* \mathbf{y}_n^* \tag{2.19}$$

For each coordinate i, $(\mathbf{x}_i^k, \mathbf{y}_i^k) \in \text{graph}(\varphi)$ for every $k = 0, 1, \dots$. Since φ is closed, $(\mathbf{x}_i^*, \mathbf{y}_i^*) \in \text{graph}(\varphi)$. That is, $\mathbf{y}_i^* \in \varphi(\mathbf{x}_i^*) = \varphi(\mathbf{x}^*)$ for every $i = 0, 1, \dots, n$. Therefore, (2.19) implies

$$\mathbf{x}^* \in \text{conv } \varphi(\mathbf{x}^*)$$

Since φ is convex valued,

$$\mathbf{x}^* \in \varphi(\mathbf{x}^*)$$

2.142 Analogous to Exercise 2.129, there exists a simplex T containing S and a retraction of T onto S, that is a continuous function $g\colon T \to S$ with $g(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in S$. Then $\varphi \circ g\colon T \rightrightarrows S \subset T$ is closed-valued (Exercise 2.106) and uhc (Exercise 2.103). By the argument in the proof, there exists a point $\mathbf{x}^* \in T$ such that $\mathbf{x}^* \in \varphi \circ g(\mathbf{x}^*)$. However, since $\varphi \circ g(\mathbf{x}^*) \subseteq S$, we must have $\mathbf{x}^* \in S$ and therefore $g(\mathbf{x}^*) = \mathbf{x}^*$. This implies $\mathbf{x}^* \in \varphi(\mathbf{x}^*)$. That is, $\mathbf{x}^*$ is a fixed point of φ.

2.143 $B = B_1 \times B_2 \times \ldots \times B_n$ is the Cartesian product of uhc, compact- and convex-valued correspondences. Therefore B is also compact-valued and uhc (Exercise 2.112 and also convex-valued (Exercise 1.165). By Exercise 2.106, B is closed.

2.144 Strict quasiconcavity ensures that the best response correspondence is in fact a function $B\colon S \to S$. Since the hypotheses of Example 2.96 apply, there exists at least one equilibrium. Suppose that there are two Nash equilibria $\mathbf{s}$ and $\mathbf{s}'$. Since B is a contraction,

$$\rho(B(\mathbf{s}), B(\mathbf{s}') \leq \beta\rho(\mathbf{s}, \mathbf{s}')$$

for some $\beta < 1$. However

$$B(\mathbf{s}) = \mathbf{s} \text{ and } \; B(\mathbf{s}') = \mathbf{s}'$$

and (2.19) implies that

$$\rho(\mathbf{s}, \mathbf{s}') \leq \beta\rho(\mathbf{s}, \mathbf{s}')$$

which is possible if and only if $\mathbf{s} = \mathbf{s}'$. This implies that the equilibrium must be unique.

2.145 Since K is compact, it is totally bounded (Exercise 1.112). There exists a finite set of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ such that

$$K \subseteq \bigcap_{i=1}^{n} B_\epsilon(\mathbf{x}_i)$$

Let $S = \operatorname{conv}\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$. For $i = 1, 2, \ldots, n$ and $\mathbf{x} \in X$, define

$$\alpha_i(\mathbf{x}) = \max\{0, \epsilon - \|\mathbf{x} - \mathbf{x}_i\|\}$$

Then for every $\mathbf{x} \in K$,

$$0 \leq \alpha_i(\mathbf{x}) \leq \epsilon, \qquad i = 1, 2, \ldots, n$$

and

$$\alpha_i(\mathbf{x}) > 0 \iff \mathbf{x} \in B_\epsilon(\mathbf{x}_i)$$

Note that $\alpha_i(\mathbf{x}) > 0$ for some i. Define

$$h(\mathbf{x}) = \frac{\sum \alpha_i(\mathbf{x})\mathbf{x}_i}{\sum \alpha_i(\mathbf{x})}$$

Then $h(\mathbf{x}) \in S$ and therefore $h\colon K \to S$. Furthermore, h is continuous and

$$\begin{aligned}
\|h(\mathbf{x}) - \mathbf{x}\| &= \left\| \frac{\sum \alpha_i(\mathbf{x})\mathbf{x}_i}{\sum \alpha_i(\mathbf{x})} - \mathbf{x} \right\| \\
&= \left\| \frac{\sum \alpha_i(\mathbf{x})(\mathbf{x}_i - \mathbf{x})}{\sum \alpha_i(\mathbf{x})} \right\| \\
&= \frac{\sum \alpha_i(\mathbf{x}) \|\mathbf{x}_i - \mathbf{x}\|}{\sum \alpha_i(\mathbf{x})} \\
&\leq \frac{\sum \alpha_i(\mathbf{x})\epsilon}{\sum \alpha_i(\mathbf{x})} = \epsilon
\end{aligned}$$

since $\alpha_i(\mathbf{x}) > 0 \iff \|\mathbf{x}_i - \mathbf{x}\| \leq \epsilon$.

2.146 1. For every $\mathbf{x} \in S^k$, $f(\mathbf{x}) \in S$ and therefore $g^k(\mathbf{x}) = h^k\big(f(\mathbf{x})\big) \in S^k$.

2. For any $\mathbf{x} \in S^k$, let $\mathbf{y} = f(\mathbf{x}) \in f(S)$ and therefore

$$\left\| h^k(\mathbf{y}) - \mathbf{y} \right\| < \frac{1}{k}$$

which implies

$$\left\| g^k(\mathbf{x}) - f(\mathbf{x}) \right\| \leq \frac{1}{k} \text{ for every } \ \mathbf{x} \in S^k$$

2.147 By the Triangle inequality

$$\left\| \mathbf{x}^k - f(\mathbf{x}) \right\| \leq \left\| g^k(\mathbf{x}^k) - f(\mathbf{x}^k) \right\| + \left\| f(\mathbf{x}^k) - f(\mathbf{x}) \right\|$$

As shown in the previous exercise

$$\left\| g^k(\mathbf{x}^k) - f(\mathbf{x}^k) \right\| \leq \frac{1}{k} \to 0$$

as $k \to \infty$. Also since f is continuous

$$\left\| f(\mathbf{x}^k) - f(\mathbf{x}) \right\| \to 0$$

Therefore

$$\left\| \mathbf{x}^k - f(\mathbf{x}) \right\| \to 0 \implies \mathbf{x} = f(\mathbf{x})$$

$\mathbf{x}$ is a fixed point of f.

2.148 $T(F)$ is bounded and equicontinuous and so therefore is $\overline{T(F)}$ (Exercise 2.96). By Ascoli's theorem (Exercise 2.95), $\overline{T(F)}$ is compact. Therefore T is a compact operator. Applying Corollary 2.8.1, T has a fixed point.