Solutions Manual
Foundations of Mathematical Economics

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Chapter 1: Sets and Spaces

1.1

$$\{1, 3, 5, 7 \dots\} \text{ or } \{n \in N : n \text{ is odd}\}$$

1.2 Every $x \in A$ also belongs to B. Every $x \in B$ also belongs to A. Hence A, B have precisely the same elements.

1.3 Examples of finite sets are

- the letters of the alphabet { A, B, C, ..., Z }
- the set of consumers in an economy
- the set of goods in an economy
- the set of players in a game.

Examples of infinite sets are

- the real numbers $\Re$
- the natural numbers $\mathfrak{N}$
- the set of all possible colors
- the set of possible prices of copper on the world market
- the set of possible temperatures of liquid water.

1.4 $S = \{1, 2, 3, 4, 5, 6\}$, $E = \{2, 4, 6\}$.

1.5 The player set is $N = \{\text{Jenny}, \text{Chris}\}$. Their action spaces are

$$A_i = \{\text{Rock}, \text{Scissors}, \text{Paper}\} \qquad i = \text{Jenny}, \text{Chris}$$

1.6 The set of players is $N = \{1, 2, \dots, n\}$. The strategy space of each player is the set of feasible outputs

$$A_i = \{q_i \in \Re_+ : q_i \leq Q_i\}$$

where q_i is the output of dam i.

1.7 The player set is $N = \{1, 2, 3\}$. There are $2^3 = 8$ coalitions, namely

$$\mathcal{P}(N) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

There are 2^{10} coalitions in a ten player game.

1.8 Assume that $x \in (S \cup T)^c$. That is $x \notin S \cup T$. This implies $x \notin S$ and $x \notin T$, or $x \in S^c$ and $x \in T^c$. Consequently, $x \in S^c \cap T^c$. Conversely, assume $x \in S^c \cap T^c$. This implies that $x \in S^c$ and $x \in T^c$. Consequently $x \notin S$ and $x \notin T$ and therefore $x \notin S \cup T$. This implies that $x \in (S \cup T)^c$. The other identity is proved similarly.

1.9

$$\bigcup_{S \in \mathcal{C}} S = N$$

$$\bigcap_{S \in \mathcal{C}} S = \emptyset$$

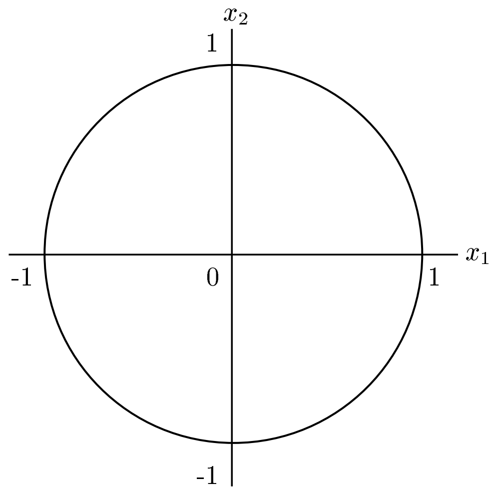

Figure 1.1: The relation $\{ (x, y) : x^2 + y^2 = 1 \}$

1.10 The sample space of a single coin toss is $\{ H, T \}$. The set of possible outcomes in three tosses is the product

$$\begin{aligned}\{H, T\} \times \{H, T\} \times \{H, T\} = \{ &(H, H, H), (H, H, T), (H, T, H),\\ &(H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T) \}\end{aligned}$$

A typical outcome is the sequence (H, H, T) of two heads followed by a tail.

1.11

$$Y \cap \Re^n_+ = \{\mathbf{0}\}$$

where $\mathbf{0} = (0, 0, \dots, 0)$ is the production plan using no inputs and producing no outputs. To see this, first note that $\mathbf{0}$ is a feasible production plan. Therefore, $\mathbf{0} \in Y$. Also, $\mathbf{0} \in \Re^n_+$ and therefore $\mathbf{0} \in Y \cap \Re^n_+$.

To show that there is no other feasible production plan in $\Re^n_+$, we assume the contrary. That is, we assume there is some feasible production plan $\mathbf{y} \in \Re^n_+ \setminus \{\mathbf{0}\}$. This implies the existence of a plan producing a positive output with no inputs. This technological infeasible, so that $y \notin Y$.

1.12
1. Let $\mathbf{x} \in V(y)$. This implies that $(y, -\mathbf{x}) \in Y$. Let $\mathbf{x}' \geq \mathbf{x}$. Then $(y, -\mathbf{x}') \leq (y, -\mathbf{x})$ and free disposability implies that $(y, -\mathbf{x}') \in Y$. Therefore $\mathbf{x}' \in V(y)$.
2. Again assume $\mathbf{x} \in V(y)$. This implies that $(y, -\mathbf{x}) \in Y$. By free disposal, $(y', -\mathbf{x}) \in Y$ for every $y' \leq y$, which implies that $\mathbf{x} \in V(y')$. $V(y') \supseteq V(y)$.

1.13 The domain of "<" is $\{1, 2\} = X$ and the range is $\{2, 3\} \subsetneqq Y$.

1.14 Figure 1.1.

1.15 The relation "is strictly higher than" is transitive, antisymmetric and asymmetric. It is not complete, reflexive or symmetric.

1.16 The following table lists their respective properties.

	$<$	$\leq$	$=$
reflexive	$\times$	$\surd$	$\surd$
transitive	$\surd$	$\surd$	$\surd$
symmetric	$\times$	$\surd$	$\surd$
asymmetric	$\surd$	$\times$	$\times$
anti-symmetric	$\surd$	$\surd$	$\surd$
complete	$\surd$	$\surd$	$\times$

Note that the properties of symmetry and anti-symmetry are not mutually exclusive.

1.17 Let $\sim$ be an equivalence relation of a set $X \neq \emptyset$. That is, the relation $\sim$ is reflexive, symmetric and transitive. We first show that every $x \in X$ belongs to some equivalence class. Let a be any element in X and let $\sim (a)$ be the class of elements equivalent to a, that is

$$\sim\!(a) \equiv \{\, x \in X : x \sim a \,\}$$

Since $\sim$ is reflexive, $a \sim a$ and so $a \in \sim\!(a)$. Every $a \in X$ belongs to some equivalence class and therefore

$$X = \bigcup_{a \in X} \sim\!(a)$$

Next, we show that the equivalence classes are either disjoint or identical, that is $\sim\!(a) \neq \sim\!(b)$ if and only if f$\sim\!(a) \cap \sim\!(b) = \emptyset$.

First, assume $\sim\!(a) \cap \sim\!(b) = \emptyset$. Then $a \in \sim\!(a)$ but $a \notin \sim\!(b)$. Therefore $\sim\!(a) \neq \sim\!(b)$.

Conversely, assume $\sim\!(a) \cap \sim\!(b) \neq \emptyset$ and let $x \in \sim\!(a) \cap \sim\!(b)$. Then $x \sim a$ and by symmetry $a \sim x$. Also $x \sim b$ and so by transitivity $a \sim b$. Let y be any element in $\sim\!(a)$ so that $y \sim a$. Again by transitivity $y \sim b$ and therefore $y \in \sim\!(b)$. Hence $\sim\!(a) \subseteq \sim\!(b)$. Similar reasoning implies that $\sim\!(b) \subseteq \sim\!(a)$. Therefore $\sim\!(a) = \sim\!(b)$.

We conclude that the equivalence classes partition X.

1.18 The set of proper coalitions is not a partition of the set of players, since any player can belong to more than one coalition. For example, player 1 belongs to the coalitions $\{1\}$, $\{1, 2\}$ and so on.

1.19

$$\begin{aligned} x \succ y &\implies x \succsim y \text{ and } y \not\succsim x \\ y \sim z &\implies y \succsim z \text{ and } z \succsim y \end{aligned}$$

Transitivity of $\succsim$ implies $x \succsim z$. We need to show that $z \not\succsim x$. Assume otherwise, that is assume $z \succsim x$ This implies $z \sim x$ and by transitivity $y \sim x$. But this implies that $y \succsim x$ which contradicts the assumption that $x \succ y$. Therefore we conclude that $z \not\succsim x$ and therefore $x \succ z$. The other result is proved in similar fashion.

1.20 **asymmetric** Assume $x \succ y$.

$$x \succ y \implies y \not\succsim x$$

while

$$y \succ x \implies y \succsim x$$

Therefore

$$x \succ y \implies y \not\succ x$$

transitive Assume $x \succ y$ and $y \succ z$.

$$\begin{aligned} x \succ y &\implies x \succsim y \text{ and } y \not\succsim x \\ y \succ z &\implies y \succsim z \text{ and } z \not\succsim y \end{aligned}$$

Since $\succsim$ is transitive, we conclude that $x \succsim z$.

It remains to show that $z \not\succsim x$. Assume otherwise, that is assume $z \succsim x$. We know that $x \succsim y$ and transitivity implies that $z \succsim y$, contrary to the assumption that $y \succ z$. We conclude that $z \not\succsim x$ and

$$x \succsim z \text{ and } z \not\succsim x \implies x \succ z$$

This shows that $\succ$ is transitive.

1.21 reflexive Since $\succsim$ is reflexive, $x \succsim x$ which implies $x \sim x$.

transitive Assume $x \sim y$ and $y \sim z$. Now

$$x \sim y \iff x \succsim y \text{ and } y \succsim x$$

$$y \sim z \iff y \succsim z \text{ and } z \succsim y$$

Transitivity of $\succsim$ implies

$$x \succsim y \text{ and } y \succsim z \implies x \succsim z$$

$$z \succsim y \text{ and } y \succsim x \implies z \succsim x$$

Combining

$$x \succsim z \text{ and } z \succsim x \implies x \sim z$$

symmetric

$$\begin{aligned} x \sim y &\iff x \succsim y \text{ and } y \succsim x \\ &\iff y \succsim x \text{ and } x \succsim y \\ &\iff y \sim x \end{aligned}$$

1.22 reflexive Every integer is a multiple of itself, that is $m = 1m$.

transitive Assume $m = kn$ and $n = lp$ where $k, l \in N$. Then $m = klp$ so that m is a multiple of p.

not symmetric If $m = kn$, $k \in N$, then $n = \frac{1}{k}m$ and $k \notin N$. For example, 4 is a multiple of 2 but 2 is not a multiple of 4.

1.23

$$\begin{aligned} [a, b] &= \{ a, y, b, z \} \\ (a, b) &= \{ y \} \end{aligned}$$

1.24

$$\begin{aligned} \succsim (y) &= \{b, y, z\} \\ \succ (y) &= \{b, z\} \\ \precsim (y) &= \{a, x, y\} \\ \prec (y) &= \{a, x\} \end{aligned}$$

1.25 Let X be ordered by $\succsim$. $x \in X$ is a minimal element there is no element which strictly precedes it, that is there is no element $y \in X$ such that $y \prec x$. $x \in X$ is the first element if it precedes every other element, that is $x \precsim y$ for all $y \in X$.

1.26 The maximal elements of X are b and z. The minimal element of X is x. These are also best and worst elements respectively.

1.27 Assume that x is a best element in X ordered by $\succsim$. That is, $x \succsim y$ for all $y \in X$. This implies that there is no $y \in X$ which strictly dominates x. Therefore, x is maximal in X. In Example 1.23, the numbers $5, 6, 7, 8, 9$ are all maximal elements, but none of them is a best element.

1.28 Assume that the elements are denoted $x_1, x_2, \dots, x_n$. We can identify the maximal element by constructing another list using the following recursive algorithm

$$a_1 = x_1$$
$$a_i = \begin{cases} x_i & \text{if } x_i \succ a_{i-1} \\ a_{i-1} & \text{otherwise} \end{cases}$$

By construction, there is no x_i which strictly succedes a_n. a_n is a maximal element.

1.29

$$x^* \text{ is maximal} \iff \text{there does not exist } x \succ x^*$$

that is

$$\succ(x^*) = \{\, x : x \succ x^* \,\} = \emptyset$$

$$\begin{aligned} x^* \text{ is best} &\iff x^* \succsim x \text{ for every } \; x \in X \\ &\iff x \precsim x^* \text{ for every } \; x \in X \end{aligned}$$

That is, every $x \in X$ belongs to $\precsim(x^*)$ or $\precsim(x^*) = X$.

1.30 Let A be a nonempty set of a set X ordered by $\succsim$. $x \in X$ is a lower bound for A if it precedes every element in A, that is $x \precsim a$ for all $a \in A$. It is a greatest lower bound if it dominates every lower bound, that is $x \succsim y$ for every lower bound y of A.

1.31 Any multiple of 60 is an upper bound for A. Thus, the set of upper bounds of A is $\{60, 120, 240, \dots\}$. The least upper bound of A is 60. The only lower bound is 1, hence it is the greatest lower bound.

1.32 The least upper bounds of interval $[a, b]$ are b and z. The least upper bound of (a, b) is y.

1.33

$$\begin{aligned} x \text{ is an upper bound of } A &\iff x \succsim a \text{ for every } \; a \in A \\ &\iff a \precsim x \text{ for every } \; a \in A \\ &\iff A \subseteq \precsim(x) \end{aligned}$$

Similarly

$$\begin{aligned} x \text{ is a lower bound of } A &\iff x \precsim a \text{ for every } \; a \in A \\ &\iff a \succsim x \text{ for every } \; a \in A \\ &\iff A \subseteq \succsim(x) \end{aligned}$$

1.34 For every $x \in \Re^2$,

$$x \succ y \text{ if } x_1 > y_1 \text{ or } x_1 = y_1 \text{ and } x_2 > y_2$$

Since all elements $x \in \Re^2$ are comparable, $\succ$ is complete; it is a total order.

1.35 Assume $\succsim_i$ is complete for every i. Then for every $x, y \in X$ and for all $i = 1, 2, \dots, n$, either $x_i \succsim_i y_i$ or $y_i \succsim_i x_i$ or both. Either

$x_i \sim_i y_i$ **for all** i Then define $x \sim y$.

$x_i \not\sim_i y_i$ **for some** i Let k be the first individual with a strict preference, that is $k = \min_i(x_i \not\sim y_i)$. (Completeness of $\succsim_i$ ensures that k is defined). Then define

$$x \succ y \text{ if } x_k \succ_i y_k$$
$$y \succ x \text{ otherwise}$$

1.36 Let S, T and U be subsets of a finite set X. Set inclusion $\subseteq$ is

reflexive since $S \subseteq S$.

transitive since $S \subseteq T$ and $T \subseteq U$ implies $S \subseteq U$.

anti-symmetric since $S \subseteq T$ and $T \subseteq S$ implies $S = T$

Therefore $\subseteq$ is a partial order.

1.37 Assume x and y are both least upper bounds of A. That is $x \succsim a$ for all $a \in A$ and $y \succsim a$ for all $a \in A$. Further, if x is a least upper bound, $y \succsim x$. If y is a least upper bound, $x \succsim y$. By anti-symmetry, $x = y$.

1.38

$$x \sim y \implies x \succsim y \text{ and } y \succsim x$$

which implies that $x = y$ by antisymmetry. Each equivalence class

$$\sim (x) = \{\, y \in X : y \sim x \,\}$$

comprises just a single element x.

1.39 $\max \mathcal{P}(X) = X$ and $\min \mathcal{P}(X) = \emptyset$.

1.40 The subset $\{2, 4, 8\}$ forms a chain. More generally, the set of integer powers of a given number $\{\, n, n^2, n^3, \dots \,\}$ forms a chain.

1.41 Assume x and y are maximal elements of the chain A. Then $x \succsim a$ for all $a \in A$ and in particular $x \succsim y$. Similarly, $y \succsim a$ for all $a \in A$ and in particular $y \succsim x$. Since $\succsim$ is anti-symmetric, $x = y$.

1.42

1. By assumption, for every $t \in T \setminus W$, $\prec(t)$ is a nonempty finite chain. Hence, it has a unique maximal element, $p(t)$.
2. Let t be any node. Either t is an initial node or t has a unique predecessor $p(t)$. Either $p(t)$ is an initial node, or it has a unique predecessor $p(p(t))$. Continuing in this way, we trace out a unique path from t back to an initial node. We can be sure of eventually reaching an initial node since T is finite.

1.43

$$(1, 2) \vee (3, 1) = (3, 2) \text{ and } (1, 2) \wedge (3, 2) = (1, 2)$$

1.44 1. $x \vee y$ is an upper bound for $\{x, y\}$, that is $\mathbf{x} \vee \mathbf{y} \succsim x$ and $\mathbf{x} \vee \mathbf{y} \succsim y$. Similarly, $x \vee y$ is a lower bound for $\{x, y\}$.

2. Assume $x \succsim y$. Then x is an upper bound for $\{x, y\}$, that is $x \succsim x \vee y$. If b is any upper bound for $\{x, y\}$, then $b \succsim x$. Therefore, x is the least upper bound for $\{x, y\}$. Similarly, y is a lower bound for $\{x, y\}$, and is greater than any other lower bound. Conversely, assume $x \vee y = x$. Then x is an upper bound for $\{x, y\}$, that is $x \succsim y$.

3. Using the preceding equivalence

$$x \succsim x \wedge y \implies x \vee (x \wedge y) = x$$
$$x \vee y \succsim x \implies (x \vee y) \wedge x = x$$

1.45 A chain X is a complete partially ordered set. For every $x, y \in X$ with $x \neq y$, either $x \succ y$ or $y \succ x$. Therefore, define the meet and join by

$$x \wedge y = \begin{cases} y & \text{if } x \succ y \\ x & \text{if } y \succ x \end{cases}$$

$$x \vee y = \begin{cases} x & \text{if } x \succ y \\ y & \text{if } y \succ x \end{cases}$$

X is a lattice with these operations.

1.46 Assume X_1 and X_2 are lattices, and let $X = X_1 \times X_2$. Consider any two elements $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ in X. Since X_1 and X_2 are lattices, $b_1 = x_1 \vee y_1 \in X_1$ and $b_2 = x_2 \vee y_2 \in X_2$, so that $\mathbf{b} = (b_1, b_2) = (x_1 \vee y_1, x_2 \vee y_2) \in X$. Furthermore $\mathbf{b} \succsim \mathbf{x}$ and $\mathbf{b} \succsim \mathbf{y}$ in the natural product order, so that $\mathbf{b}$ is an upper bound for the $\{\mathbf{x}, \mathbf{y}\}$. Every upper bound $\hat{\mathbf{b}} = (\hat{b}_1, \hat{b}_2)$ of $\{\mathbf{x}, \mathbf{y}\}$ must have $b_i \succsim_i x_i$ and $b_i \succsim_i y_i$, so that $\hat{\mathbf{b}} \succsim \mathbf{b}$. Therefore, $\mathbf{b}$ is the least upper bound of $\{\mathbf{x}, \mathbf{y}\}$, that is $\mathbf{b} = \mathbf{x} \vee \mathbf{y}$. Similarly, $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, x_2 \wedge y_2)$.

1.47 Let S be a subset of X and let

$$S^* = \{x \in X : x \succsim s \text{ for every } \ s \in S\}$$

be the set of upper bounds of S. Then $x^* \in S^* \neq \emptyset$. By assumption, S^* has a greatest lower bound b. Since every $s \in S$ is a lower bound of S^*, $b \succsim s$ for every $s \in S$. Therefore b is an upper bound of S. Furthermore, b is the least upper bound of S, since $b \precsim x$ for every $x \in S^*$. This establishes that every subset of X also has a least upper bound. In particular, every pair of elements has a least upper and a greatest lower bound. Consequently X is a complete lattice.

1.48 Without loss of generality, we will prove the closed interval case. Let $[a, b]$ be an interval in a lattice L. Recall that $a = \inf[a, b]$ and $b = \sup[a, b]$. Choose any x, y in $[a, b] \subseteq L$. Since L is a lattice, $x \vee y \in L$ and

$$x \vee y = \sup\{x, y\} \precsim b$$

Therefore $x \vee y \in [a, b]$. Similarly, $x \wedge y \in [a, b]$. $[a, b]$ is a lattice. Similarly, for any subset $S \subseteq [a, b] \subseteq L$, $\sup S \in L$ if L is complete. Also, $\sup S \precsim b = \sup[a, b]$. Therefore $\sup S \in [a, b]$. Similarly $\inf S \in [a, b]$ so that $[a, b]$ is complete.

1.49 1. The strong set order $\succsim_S$ is

antisymmetric Let $S_1, S_2 \subseteq X$ with $S_1 \succsim_S S_2$ and $S_2 \succsim_S S_1$. Choose $x_1 \in S_1$ and $x_2 \in S_2$. Since $S_1 \succsim_S S_2$, $x_1 \vee x_2 \in S_1$ and $x_1 \wedge x_2 \in S_2$. On the other hand, since $S_2 \succsim S_1$, $x_1 = (x_1 \vee (x_1 \wedge x_2) \in S_2$ and $x_2 = x_2 \wedge (x_1 \vee x_2) \in S_1$ (Exercise 1.44. Therefore $S_1 = S_2$ and $\succsim_S$ is antisymmetric.

transitive Let $S_1, S_2, S_3 \subseteq X$ with $S_1 \succsim_S S_2$ and $S_2 \succsim_S S_3$. Choose $x_1 \in S_1$, $x_2 \in S_2$ and $x_3 \in S_3$. Since $S_1 \succsim_S S_2$ and $S_2 \succsim_S S_3$, $x_1 \vee x_2$ and $x_2 \wedge x_3$ are in S_2. Therefore $y_2 = x_1 \vee (x_2 \wedge x_3) \in S_2$ which implies

$$\begin{aligned} x_1 \vee x_3 &= x_1 \vee \big((x_2 \wedge x_3) \vee x_3\big) \\ &= \big(x_1 \vee (x_2 \wedge x_3)\big) \vee x_3 \\ &= y_2 \vee x_3 \in S_3 \end{aligned}$$

since $S_2 \succsim_S S_3$. Similarly $z_2 = (x_1 \vee x_2) \wedge x_3 \in S_2$ and

$$\begin{aligned} x_1 \wedge x_3 &= \big(x_1 \wedge (x_1 \vee x_2)\big) \wedge x_3 \\ &= x_1 \wedge \big((x_1 \vee x_2) \wedge x_3\big) \\ &= x_1 \wedge z_2 \in S_1 \end{aligned}$$

Therefore, $S_1 \succsim_S S_3$.

2. $S \succsim_S S$ if and only if, for every $x_1, x_2 \in S$, $x_1 \vee x_2 \in S$ and $x_1 \wedge x_2 \in S$, which is the case if and only if S is a sublattice.

3. Let $L(X)$ denote the set of all sublattices of X. We have shown that $\succsim_S$ is reflexive, transitive and antisymmetric on $L(X)$. Hence, it is a partial order on $L(X)$.

1.50 Assume $S_1 \succsim_S S_2$. For any $x_1 \in S_1$ and $x_2 \in S_2$, $x_1 \vee x_2 \in S_1$ and $x_1 \wedge x_2 \in S_2$. Therefore

$$\sup S_1 \succsim x_1 \vee x_2 \succsim x_2 \qquad \text{for every } \; x_2 \in S_2$$

which implies that $\sup S_1 \succsim \sup S_2$. Similarly

$$\inf S_2 \precsim x_1 \wedge x_2 \precsim x_1 \qquad \text{for every } \; x_1 \in S_1$$

which implies that $\inf S_2 \precsim \inf S_1$. Note that completeness ensures the existence of $\sup S$ and $\inf S$ respectively.

1.51 An argument analogous to the preceding exercise establishes $\boxed{\Longrightarrow}$. (Completeness is not required, since for any interval $a = \inf[a, b]$ and $b = \sup[a, b]$).

To establish the converse, assume that $S_1 = [a_1, b_1]$ and $S_2 = [a_2, b_2]$. Consider any $x_1 \in S_1$ and $x_2 \in S_2$. There are two cases.

Case 1. $x_1 \succsim x_2$ Since X is a chain, $x_1 \vee x_2 = x_1 \in S_1$. $x_1 \wedge x_2 = x_2 \in S_2$.

Case 2. $x_1 \prec x_2$ Since X is a chain, $x_1 \vee x_2 = x_2$. Now $a_1 \precsim x_1 \prec x_2 \precsim b_2 \precsim b_2$. Therefore, $x_2 = x_1 \vee x_2 \in S_1$. Similarly $a_2 \precsim a_1 \precsim x_1 \prec x_2 \precsim b_2$. Therefore $x_1 \wedge x_2 = x_1 \in S_2$.

We have shown that $S_1 \succsim_S S_2$ in both cases.

1.52 Assume that $\succsim$ is a complete relation on X. This means that for every $x, y \in X$, either $x \succsim y$ or $y \succsim x$. In particular, letting $x = y$, $x \succsim x$ for $x \in X$. $\succsim$ is reflexive.

1.53 Anti-symmetry implies that each indifference class contains a single element. If the consumer's preference relation was anti-symmetric, there would be no baskets of goods between which the consumer was indifferent. Each indifference curve which consist a single point.

1.54 We previously showed (Exercise 1.27) that every best element is maximal. To prove the converse, assume that x is maximal in the weakly ordered set X. We have to show that $x \succsim y$ for all $y \in X$. Assume otherwise, that is assume there is some $y \in X$ for which $x \not\succsim y$. Since $\succsim$ is complete, this implies that $y \succ x$ which contradicts the assumption that x is maximal. Hence we conclude that $x \succsim y$ for $y \in X$ and x is a best element.

1.55 False. A chain has at most one maximal element (Exercise 1.41). Here, uniqueness is ensured by anti-symmetry. A weakly ordered set in which the order is not anti-symmetric may have multiple maximal and best elements. For example, a and b are both best elements in the weakly ordered set $\{a \sim b \succ c\}$.

1.56
1. For every $x \in X$, either $x \succsim y \implies x \in \succsim(y)$ or $y \succsim x \implies x \in \precsim(y)$ since $\succsim$ is complete. Consequently, $\succsim(y) \cup \prec(y) = X$ If $x \in \succsim(y) \cap \precsim(y)$, then $x \succsim y$ and $y \succsim x$ so that $x \sim y$ and $x \in I_y$.
2. For every $x \in X$, either $x \succsim y \implies x \in \succsim(y)$ or $y \succ x \implies x \in \prec(y)$ since $\succsim$ is complete. Consequently, $\succsim(y) \cup \prec(y) = X$ and $\succsim(y) \cap \prec(y) = \emptyset$.
3. For every $y \in X$, $\succ(y)$ and I_y partition $\succsim(y)$ and therefore $\succ(y)$, I_y and $\prec(y)$ partition X.

1.57 Assume $x \succsim y$ and $z \in \succsim(x)$. Then $z \succsim x \succsim y$ by transitivity. Therefore $z \in \succsim(y)$. This shows that $\succsim(x) \subseteq \succsim(y)$.

Similarly, assume $x \succ y$ and $z \in \succ(x)$. Then $z \succ x \succ y$ by transitivity. Therefore $z \in \succ(y)$. This shows that $\succsim(x) \subseteq \succsim(y)$. To show that $\succsim(x) \neq \succsim(y)$, observe that $x \in \succ(y)$ but that $x \notin \succ(x)$

1.58 Every finite ordered set has a least one maximal element (Exercise 1.28).

1.59 Kreps (1990, p.323), Luenberger (1995, p.170) and Mas-Colell et al. (1995, p.313) adopt the weak Pareto order, whereas Varian (1992, p.323) distinguishes the two orders. Osborne and Rubinstein (1994, p.7) also distinguish the two orders, utilizing the weak order in defining the core (Chapter 13) but the strong Pareto order in the Nash bargaining solution (Chapter 15).

1.60 Assume that a group S is decisive over $x, y \in X$. Let $a, b \in X$ be two other states. We have to show that S is decisive over a and b. Without loss of generality, assume for all individuals $a \succsim_i x$ and $y \succsim_i b$. Then, the Pareto order implies that $a \succ x$ and $y \succ b$.

Assume that for every $i \in S$, $x \succsim_i y$. Since S is decisive over x and y, the social order ranks $x \succsim y$. By transitivity, $a \succsim b$. By IIA, this holds irrespective of individual preferences on other alternatives. Hence, S is decisive over a and b.

1.61 Assume that S is decisive. Let x, y and z be any three alternatives and assume $x \succsim y$ for every $i \in S$. Partition S into two subgroups S_1 and S_2 so that

$$x \succsim_i z \text{ for every } \ i \in S_1 \text{ and } \ z \succsim_i y \text{ for every } \ i \in S_2$$

Since S is decisive, $x \succsim y$. By completeness, *either*

$x \succsim z$ in which case S_1 is decisive over x and z. By the field expansion lemma (Exercise 1.60), S_1 is decisive.

$z \succ x$ which implies $z \succsim y$. In this case, S_2 is decisive over y and z, and therefore (Exercise 1.60) decisive.

1.62 Assume $\succ$ is a social order which is Pareto and satisfies Independence of Irrelevant Alternatives. By the Pareto principle, the whole group is decisive over any pair of alternatives. By the previous exercise, some proper subgroup is decisive. Continuing in this way, we eventually arrive at a decisive subgroup of one individual. By the Field Expansion Lemma (Exercise 1.60), that individual is decisive over every pair of alternatives. That is, the individual is a dictator.

1.63 Assume A is decisive over x and y and B is decisive over w and z. That is, assume

$$\begin{aligned} x \succ_A y &\implies x \succ y \\ w \succ_B z &\implies w \succ z \end{aligned}$$

Also assume

$$\begin{aligned} y \succsim_i w & \qquad \text{for every } i \\ z \succsim_i x & \qquad \text{for every } i \end{aligned}$$

This implies that $y \succsim w$ and $z \succsim x$ (Pareto principle). Combining these preferences, transitivity implies that

$$x \succ y \succsim w \succ z$$

which contradicts the assumption that $z \succsim x$. Therefore, the implied social ordering is intransitive.

1.64 Assume $x \in$ core. In particular this implies that there does not exist any $y \in W(N)$ such that $y \succ x$. Therefore $x \in$ Pareto.

1.65 No state will accept a cost share which exceeds what it can achieve on its own, so that if $x \in$ core then

$$\begin{aligned} x_{AP} &\leq 1870 \\ x_{TN} &\leq 5330 \\ x_{AP} &\leq 860 \end{aligned}$$

Similarly, the combined share of the two states AP and TN should not exceed 6990, which they could achieve by proceeding without KM, that is

$$x_{AP} + x_{TN} \leq 6990$$

Similarly

$$\begin{aligned} x_{AP} + x_{KM} &\leq 1960 \\ x_{TN} + x_{KM} &\leq 5020 \end{aligned}$$

Finally, the sum of the shares should equal the total cost

$$x_{AP} + x_{TN} + x_{KM} = 6530$$

The core is the set of all allocations of the total cost which satisfy the preceding inequalities.

For example, the allocation $(x_{AP} = 1500, x_{TN} = 5000, x_{KM} = 30)$ does not belong to the core, since TN and KM will object to their combined share of 5030; since they can meet their needs jointly at a total cost of 5020. One the other hand, no group can object to the allocation $(x_{AP} = 1510, x_{TN} = 5000, x_{KM} = 20)$, which therefore belongs to the core.

1.66 The usual way to model a cost allocation problem as a TP-coalitional game is to regard the potential cost savings from cooperation as the sum to be allocated. In this example, the total joint cost of 6530 represents a potential saving of 1530 over the aggregate cost of 8060 if each region goes its own way. This potential saving of 1530 measures $w(N)$. Similarly, undertaking a joint development, AP and TN could satisfy their combined requirements at a total cost of 6890. This compares with the standalone costs of 7100 (= 1870 (AP) + 5330 (TN)). Hence, the potential cost savings from their collaboration are 210 (= 7100 - 6890), which measures $w(AP,TN)$. By similar calculations, we can compute the worth of each coalition, namely

$$\begin{array}{rrr} w(AP)=0 & w(AP,TN)=210 & \\ w(TN)=0 & w(AP,KM)=770 & w(N)=1530 \\ w(KM)=0 & w(KM,TN)=1170 & \end{array}$$

An outcome in this game is an allocation of the total cost savings $w(N)=1530$ amongst the three players. This can be translated into final cost shares by subtracting each players share of the cost savings from their standalone cost. For example, a specific outcome in this game is $(x_{AP}=370, x_{TN}=330, x_{KM}=830)$, which corresponds to final cost shares of 1500 for AP, 5000 for TN and 30 for KM.

1.67 Let

$$C=\{\,\mathbf{x}\in X:\sum_{i\in S}x_i\geq w(S)\text{ for every }\ S\subseteq N\}$$

1. $\boxed{C\subseteq\text{core}}$ Assume that $\mathbf{x}\in C$. Suppose $\mathbf{x}\notin$ core. This implies there exists some coalition S and outcome $\mathbf{y}\in w(S)$ such that $\mathbf{y}\succ_i\mathbf{x}$ for every $i\in S$.
   - $\mathbf{y}\in w(S)$ implies $\sum_{i\in S}y_i\leq w(S)$ while
   - $\mathbf{y}\succ_i\mathbf{x}$ for every $i\in S$ implies $y_i>x_i$ for every $i\in S$. Summing, this implies
   $$\sum_{i\in S}y_i>\sum_{i\in S}x_i\geq w(S)$$

   This contradiction establishes that $\mathbf{x}\in$ core.

2. $\boxed{\text{core}\subseteq C}$ Assume that $\mathbf{x}\in$ core. Suppose $\mathbf{x}\notin C$. This implies there exists some coalition S such that $\sum_{i\in S}x_i<w(S)$. Let $d=w(S)-\sum_{i\in S}x_i$ and consider the allocation $\mathbf{y}$ obtained by reallocating d from S^c to S, that is
   $$y_i=\begin{cases}x_i+d/s & i\in S\\ x_i-d/(n-s) & i\notin S\end{cases}$$
   where $s=|S|$ is the number of players in S and $n=|N|$ is the number in N. Then $y_i>x_i$ for every $i\in S$ so that $\mathbf{y}\succ_i\mathbf{x}$ for every $i\in S$. Further, $\mathbf{y}\in w(S)$ since $\sum_{i\in S}y_i=\sum_{i\in S}x_i+d=w(S)$ and $\mathbf{y}\in X$ since
   $$\sum_{i\in N}y_i=\sum_{i\in S}(x_i+d/s)+\sum_{i\notin S}(x_i-d/(n-s))=\sum_{i\in N}x_i=w(N)$$
   This contradicts our assumption that $\mathbf{x}\notin$ core, establishing that $\mathbf{x}\in C$.

1.68 The 7 unanimity games for the player set $N = \{1, 2, 3\}$ are

$$u_{\{1\}}(S) = \begin{cases} 1 & \text{S = \{1\}, \{1,2\}, \{1,3\}, N} \\ 0 & \text{otherwise} \end{cases}$$

$$u_{\{2\}}(S) = \begin{cases} 1 & \text{S = \{2\}, \{1,2\}, \{2,3\}, N} \\ 0 & \text{otherwise} \end{cases}$$

$$u_{\{3\}}(S) = \begin{cases} 1 & \text{S = \{3\}, \{1,3\}, \{2,3\}, N} \\ 0 & \text{otherwise} \end{cases}$$

$$u_{\{1,2\}}(S) = \begin{cases} 1 & \text{S = \{1,2\}, N} \\ 0 & \text{otherwise} \end{cases}$$

$$u_{\{1,3\}}(S) = \begin{cases} 1 & \text{S = \{1,3\}, N} \\ 0 & \text{otherwise} \end{cases}$$

$$u_{\{2,3\}}(S) = \begin{cases} 1 & \text{S = \{2,3\}, N} \\ 0 & \text{otherwise} \end{cases}$$

$$u_N(S) = \begin{cases} 1 & \text{S = N} \\ 0 & \text{otherwise} \end{cases}$$

1.69 Firstly, consider a simple game which is a unanimity game with essential coalition T and let x be an outcome in which

$$\begin{aligned} x_i &\geq 0 && \text{for every } \; i \in T \\ x_i &= 0 && \text{for every } \; i \notin T \end{aligned}$$

and

$$\sum_{i \in N} x_i = 1$$

We claim that $x \in$ core.

Winning coalitions If S is winning coalition, then $w(S) = 1$. Furthermore, if it is a winning coalition, it must contain T, that is $T \subseteq S$ and

$$\sum_{i \in S} x_i \geq \sum_{i \in T} x_i = 1 = w(S)$$

Losing coalitions If S is a losing coalition, $w(S) = 0$ and

$$\sum_{i \in S} x_i \geq 0 = w(S)$$

Therefore $x \in$ core and so core $\neq \emptyset$.

Conversely, consider a simple game which is not a unanimity game. Suppose there exists an outcome $x \in$ core. Then

$$\sum_{i \in N} x_i w(N) = 1 \tag{1.1}$$

Since there are no veto players ($T = \emptyset$), $w(N \setminus \{i\}) = 1$ for every player $i \in N$ and

$$\sum_{j \neq i} x_j \geq w(N \setminus \{i\}) = 1$$

which implies that $x_i = 0$ for every $i \in N$ contradicting (1.1). Thus we conclude that core $= \emptyset$.

1.70 The excesses of the proper coalitions at $\mathbf{x}^1$ and $\mathbf{x}^2$ are

	$\mathbf{x}^1$	$\mathbf{x}^2$
{AP}	-180	-200
{KM}	-955	-950
{TN}	-395	-380
{AP, KM}	-365	-380
{AP, TN}	-365	-370
{KM, TN}	-180	-160

Therefore

$$d(\mathbf{x}^1) = (-180, -180, -365, -365, -395, -955)$$

and

$$d(\mathbf{x}^2) = (-160, -200, -370, -380, -380, -950)$$

$\mathbf{d}(\mathbf{x}^1) \prec^L \mathbf{d}(\mathbf{x}^2)$ which implies $\mathbf{x}^1 \succ^d \mathbf{x}^2$.

1.71 It is a weak order on X, that is $\succsim$ is reflexive, transitive and complete. Reflexivity and transitivity flow from the corresponding properties of $\succsim_L$ on $\Re^{2^n}$. Similarly, for any $\mathbf{x}, \mathbf{y} \in X$, either $\mathbf{d}(\mathbf{x}) \precsim_L \mathbf{d}(\mathbf{y})$ or $\mathbf{d}(\mathbf{y}) \precsim_L \mathbf{d}(\mathbf{x})$ since $\succsim_L$ is complete on $\Re^{2^n}$. Consequently either $\mathbf{x} \succsim \mathbf{y}$ or $\mathbf{y} \succsim \mathbf{x}$ (or both).

$\succsim$ is not a partial order since it is not antisymmetric

$$\mathbf{d}(\mathbf{x}) \precsim_L \mathbf{d}(\mathbf{y}) \text{ and } \mathbf{d}(\mathbf{y}) \precsim_L \mathbf{d}(\mathbf{x}) \text{ does not imply } \mathbf{x} = \mathbf{y}$$

1.72

$$d(S, \mathbf{x}) = w(S) - \sum_{i \in S} x_i$$

so that

$$d(S, \mathbf{x}) \leq 0 \iff \sum_{i \in S} x_i \geq w(S)$$

1.73 Assume to the contrary that $\mathbf{x} \in$ Nu but that $\mathbf{x} \notin$ core. Then, there exists a coalition T with a positive deficit $d(T, \mathbf{x}) > 0$. Since core $\neq \emptyset$, there exists some $\mathbf{y} \in X$ such that $d(S, \mathbf{y}) \leq 0$ for every $S \subseteq$ Nu. Consequently, $\mathbf{d}(\mathbf{y}) \prec \mathbf{d}(\mathbf{x})$ and $\mathbf{y} \succ \mathbf{x}$, so that $\mathbf{x} \notin$ Nu. This contradiction establishes that Nu $\subseteq$ core.

1.74 For player 1, $A_1 = \{C, N\}$ and

$$(C, C) \succsim_1 (C, C)$$
$$(C, C) \succsim_1 (N, C)$$

Similarly for player 2

$$(C, C) \succsim_2 (C, C)$$
$$(C, C) \succsim_2 (C, N)$$

Therefore, (C, C) satisfies the requirements of the definition of a Nash equilibrium (Example 1.51).

1.75 If $\mathbf{a}_i^*$ is the best element in $(A_i, \succsim_i')$ for every player i, then

$$(a_i^*, \mathbf{a}_{-i}) \succ_i (a_i, \mathbf{a}_{-i}) \text{ for every } \ a_i \in A_i \text{ and } \ \mathbf{a}_{-i} \in A_{-i}$$

for every $i \in N$. Therefore, $\mathbf{a}^*$ is a Nash equilibrium.

To show that it is unique, assume that $\bar{\mathbf{a}}$ is another Nash equilibrium. Then for every player $i \in N$

$$(\bar{a}_i, \bar{\mathbf{a}}_{-i}) \succsim_i (a_i, \bar{\mathbf{a}}_{-i}) \text{ for every } \ a_i \in A_i$$

which implies that $\bar{\mathbf{a}}$ is a maximal element of $\succsim_i'$. To see this, assume not. That is, assume that there exists some $\tilde{a}_i \in A_i$ such that $\tilde{a}_i \succ_i' \bar{a}_i$ which implies

$$(\tilde{a}_i, \mathbf{a}_{-i}) \succ_i (\bar{a}_i, \mathbf{a}_{-i}) \text{ for every } \ \mathbf{a}_{-i} \in A_{-i}$$

In particular

$$(\tilde{a}_i, \bar{\mathbf{a}}_{-i}) \succ_i (a_i^*, \mathbf{a}_{-i}^*)$$

which contradicts the assumption that $\mathbf{a}^*$ is a Nash equilibrium. Therefore, $\bar{\mathbf{a}}$ is another Nash equilibrium, then $\bar{a}_i$ is maximal in $\succsim_i'$ and hence also a best element of $\succsim_i'$ (Exercise 1.54), which contradicts the assumption that a_i^* is the unique best element. Consequently, we conclude that $\mathbf{a}^*$ is the unique Nash equilibrium of the game.

1.76 We show that $\rho(x, y) = |x - y|$ satisfies the requirements of a metric, namely

1. $|x - y| \geq 0$.
2. $|x - y| = 0$ if and only if $x = y$.
3. $|x - y| = |y - x|$.

To establish the triangle inequality, we can consider various cases. For example, if $x \leq y \leq z$

$$|x - z| + |z - y| \geq |x - z| = z - x \geq y - x = |x - y|$$

If $x \leq z \leq y$

$$|x - z| + |z - y| = z - x + y - z = y - x = |x - y|$$

and so on.

1.77 We show that $\rho_\infty x, y = \max_{i=1}^n |x_i - y_i|$ satisfies the requirements of a metric, namely

1. $\max_{i=1}^n |x_i - y_i| \geq 0$
2. $\max_{i=1}^n |x_i - y_i| = 0$ if and only if $x_i = y_i$ for all i.
3. $\max_{i=1}^n |x_i - y_i| = \max_{i=1}^n |y_i - x_i|$
4. For every i, $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$ from previous exercise. Therefore

$$\begin{aligned}\max |x_i - y_i| &\leq \max\left(|x_i - z_i| + |z_i - y_i|\right) \\ &\leq \max |x_i - z_i| + \max |z_i - y_i|\end{aligned}$$

1.78 For any n, any neighborhood of $1/n$ contains points of S (namely $1/n$) and points not in S ($1/n + \epsilon$). Hence every point in S is a boundary point. Also, 0 is a boundary point. Therefore $\mathrm{b}(S) = S \cup \{0\}$. Note that $S \subset \mathrm{b}(S)$. Therefore, S has no interior points.

1.79 1. Let $x \in \text{int}\, S$. Thus S is a neighborhood of x. Therefore, $T \supseteq S$ is a neighborhood of x, so that x is an interior point of T.

2. Clearly, if $x \in S$, then $x \in T \subseteq \overline{T}$. Therefore, assume $x \in \overline{S} \setminus S$ which implies that x is a boundary point of S. Every neighborhood of x contains other points of $S \subseteq T$. Hence $x \in \overline{T}$.

1.80 Assume that S is open. Every $x \in S$ has a neighborhood which is disjoint from S^c. Hence no $x \in S$ is a closure point of S^c. S^c contains all its closure points and is therefore closed.

Conversely, assume that S is closed. Let x be a point its complement S^c. Since S is closed and $x \notin S$, x is not a boundary point of S. This implies that x has a neighborhood N which is disjoint from S, that is $N \subseteq S^c$. Hence, x is an interior point of S^c. This implies that S^c contains only interior points, and hence is open.

1.81 Clearly x is a neighborhood of every point $x \in X$, since $B_r(x) \subseteq X$ for every $r > 0$. Hence, every point $x \in X$ is an interior point of x. Similarly, every point $x \in \emptyset$ is an interior point (there are none). Since x and $\emptyset$ are open, there complements $\emptyset$ and x are closed.

Alternatively, $\emptyset$ has no boundary points, and is therefore is open. Trivialy, on the other hand, $\emptyset$ contains all its boundary points, and is therefore closed.

1.82 Let X be a metric space. Assume X is the union of two disjoint closed sets A and B, that is

$$X = A \cup B \qquad A \cap B = \emptyset$$

Then $A = B^c$ is open as is $B = A^c$. Therefore X is not connected.

Conversely, assume that X is not connected. Then there exist disjoint open sets A and B such that $X = A \cup B$. But $A = B^c$ is also closed as is $B = A^c$. Therefore X is the union of two disjoint closed sets.

1.83 Assume S is both open and closed, $\emptyset \subset S \subset X$. We show that we can represent X as the union of two disjoint open sets, S and S^c. For any $S \subset X$, $X = S \cup S^c$ and $S \cap S^c = \emptyset$. S is open by assumption. It complement S^c is open since S is closed. Therefore, X is not connected.

Conversely, assume that S is not connected. That is, there exists two disjoint open sets S and T such that $X = S \cup T$. Now $S = T^c$, which implies that S is closed since T is open. Therefore S is both open and closed.

1.84 Assume that S is both open and closed. Then so is S^c and X is the disjoint union of two closed sets

$$x = S \cup S^c$$

so that

$$\text{b}(S) = \overline{S} \cap \overline{S^c} = S \cap S^c = \emptyset$$

Conversely, assume that $\text{b}(S) = \overline{S} \cap \overline{S^c} = \emptyset$. This implies that Consider any $x \in \overline{S}$. Since $\overline{S} \cap \overline{S^c} = \emptyset$, $x \notin \overline{S^c}$. *A fortiori*, $\mathbf{x} \notin S^c$ which implies that $x \in S$ and therefore $\overline{S} \subseteq S$. S is closed. Similarly we can show that $\overline{S^c} \subseteq S^c$ so that S^c is closed and therefore S is open. S is both open and closed.

1.85 1. Let $\{G_i\}$ be a (possibly infinite) collection of open sets. Let $G = \cup_i G_i$. Let x be a point in G. Then there exists some particular G_j which contains x. Since G_j is open, G_j is a neighborhood of x. Since $G_j \subseteq G$, x is an interior point of G. Since x is an arbitrary point in G, we have shown that every $x \in G$ is an interior point. Hence, G is open.

What happens if every G_i is empty? In this case, $G = \emptyset$ and is open (Exercise 1.81). The other possibility is that the collection $\{G_i\}$ is empty. Again $G = \emptyset$ which is open.

Suppose $\{ G_1, G_2, \dots, G_n \}$ is a finite collection of open sets. Let $G = \cap_i G_i$. If $G = \emptyset$, then it is trivially open. Otherwise, let x be a point in G. Then $x \in G_i$ for all $i = 1, 2, \dots, n$. Since the sets G_i are open, for every i, there exists an open ball $B(x, r_i) \subseteq G_i$ about x. Let r be the smallest radius of these open balls, that is $r = \min\{ r_1, r_2, \dots, r_n \}$. Then $B_r(x) \subseteq B(x, r_i)$, so that $B_r(x) \subseteq G_i$ for all i. Hence $B_r(x) \subseteq G$. x is an interior point of G and G is open.

To complete the proof, we need to deal with the trivial case in which the collection is empty. In that case, $G = \cap_i G_i = X$ and hence is open.

2. The corresponding properties of closed sets are established analogously.

1.86 1. Let x_0 be an interior point of S. This implies there exists an open ball $B \subseteq S$ about x_0. Every $x \in B$ is an interior point of S. Hence $B \subseteq \operatorname{int} S$. x_0 is an interior point of $\operatorname{int} S$ which is therefore open.

Let G be any open subset of S and x be a point in G. G is neighborhood of x, which implies that $S \supseteq G$ is also neighborhood of x. Therefore x is an interior point of S. Therefore $\operatorname{int} S$ contains every open subset $G \subseteq S$, and hence is the largest open set in S.

2. Let $\overline{\overline{S}}$ denote the closure of the set S. Clearly, $\overline{S} \subseteq \overline{\overline{S}}$. To show the converse, let x be a closure point of $\overline{S}$ and let N be a neighborhood of x. Then N contains some other point $x' \neq X$ which is a closure point of S. N is a neighborhood of x' which intersects S. Hence x is a closure point of S.

Consequently $\overline{\overline{S}} = \overline{S}$ which implies that $\overline{S}$ is closed.

Assume F is a closed subset of containing S. Then

$$\overline{S} \subseteq \overline{F} = F$$

since F is closed. Hence, $\overline{S}$ is a subset of every closed set containing S.

1.87 Every $x \in S$ is either an interior point or a boundary point. Consequently, the interior of S is the set of all $x \in S$ which are not boundary points

$$\operatorname{int} S = S \setminus \mathrm{b}(S)$$

1.88 Assume that S is closed, that is

$$\overline{S} = S \cup \mathrm{b}(S) = S$$

This implies that $\mathrm{b}(S) \subseteq S$. S contains its boundary.

Assume that S contains its boundary, that is $S \supseteq \mathrm{b}(S)$. Then

$$\overline{S} = S \cup \mathrm{b}(S) = S$$

S is closed.

1.89 Assume S is bounded, and let $d = d(S)$. Choose any $x \in S$. For all $y \in S$, $\rho(x, y) \leq d < d + 1$. Therefore, $y \in B(x, d + 1)$. S is contained in the open ball $B(x, d + 1)$.

Conversely, assume S is contained in the open ball $B_r(x)$. Then for any $y, z \in S$

$$\rho(y, z) \leq \rho(y, x) + \rho(x, z) < 2r$$

by the triangle inequality. Therefore $d(S) < 2r$ and the set is bounded.

1.90 Let $y \in S \cap B_r(x_0)$. For every $x \in S$, $\rho(x, y) < r$ and therefore

$$\rho(x, x_0) \leq \rho(x, y) + \rho(y, x_0) < r + r = 2r$$

so that $x \in B_{2r}(x_0)$.

1.91 Let $\mathbf{y}^0 \in Y$. For any $r > 0$, let $\mathbf{y}' = \mathbf{y} - r$ be the production plan which is r units less in every commodity. Then, for any $\mathbf{y} \in B_r(\mathbf{y}')$

$$y_i - y_i' \leq \rho_\infty(\mathbf{y}, \mathbf{y}') < r \qquad \text{for every } \ i$$

and therefore $\mathbf{y} < \mathbf{y}_0$. Thus $B_r(\mathbf{y}') \subset Y$ and so $\mathbf{y}' \in \operatorname{int} Y \neq \emptyset$.

1.92 For any $x \in S_1$

$$\rho_x = \rho(x, S_2) > 0$$

Similarly, for every $y \in S_2$

$$\rho_y = \rho(y, S_1) > 0$$

Let

$$T_1 = \bigcup_{x \in S_1} B_{\rho_x/2}(x)$$
$$T_2 = \bigcup_{y \in S_2} B_{\rho_y/2}(x)$$

Then T_1 and T_2 are open sets containing S_1 and S_2 respectively.

To show that T_1 and T_2 are disjoint, suppose to the contrary that $z \in T_1 \cap T_2$. Then, there exist points $x \in S_1$ and $y \in S_2$ such that

$$\rho(x, z) < \rho_x/2, \qquad \rho(y, z) < \rho_y/2$$

Without loss of generality, suppose that $\rho_x \leq \rho_y$ and therefore

$$\rho(x, y) \leq \rho(x, z) + \rho(y, z) < \rho_x/2 + \rho_y/2 \leq \rho_y$$

which contradicts the definition of ρ_y and shows that $T_1 \cap T_2 = \emptyset$.

1.93 By Exercise 1.92, there exist disjoint open sets T_1 and T_2 such that $S_1 \subseteq T_1$ and $S_2 \subseteq T_2$. Since $S_2 \subseteq T_2$, $S_2 \cap T_2^c = \emptyset$. T_2^c is a closed set which contains $\overline{T_1}$, and therefore $S_2 \cap \overline{T_1} = \emptyset$. $T = T_1$ is the desired set.

1.94 See Figure 1.2.

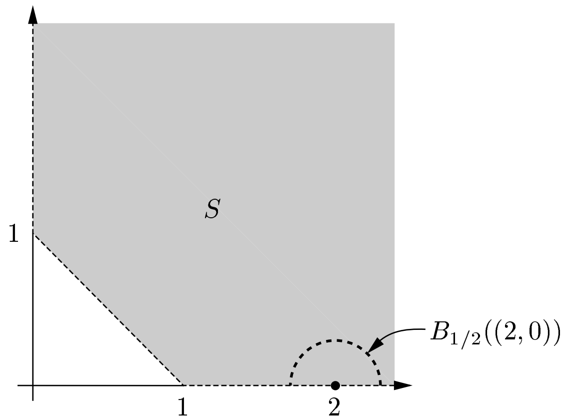

Figure 1.2: Open ball about $(2,0)$ relative to X

1.95 Assume S is connected. Suppose S is not an interval. This implies that there exists numbers x, y, z such that $x < y < z$ and $x, z \in S$ while $y \notin S$. Then

$$S = (S \cap (-\infty, y)) \cup (S \cap (y, \infty))$$

represents S as the union of two disjoint open sets (relative to S), contradicting the assumption that S is connected.

Conversely, assume that S is an interval. Suppose that S is not connected. That is, $S = A \cup B$ where A and B are nonempty disjoint closed sets. Choose $x \in A$ and $z \in B$. Since A and B are disjoint, $x \neq z$. Without loss of generality, we may assume $x < z$. Since S is an interval, $[x, z] \subseteq S = A \cup B$. Let

$$y = \sup\{\, [x, z] \cap S \,\}$$

Clearly $x \leq y \leq z$ so that $y \in S$. Now y belongs to either A or B. Since A is closed in S, $[x, z] \cap A$ is closed and $y = \sup\{\, [x, z] \cap S \,\} \in A$. This implies the $y < z$. Consequently, $y + \epsilon \in B$ for every $\epsilon > 0$ such that $y + \epsilon \leq z$. Since B is closed, $y \in B$. This implies that y belongs to both A and B contradicting the assumption that $A \cap B = \emptyset$. We conclude that S must be connected.

1.96 Assume $x^n \to x$ and also $x^n \to y$. We have to show that $x = y$. Suppose not, that is suppose $x \neq y$ (see Figure 1.3). Then $\rho(x, y) = R > 0$. Let $r = R/3 > 0$. Since $x^n \to x$, there exists some N_x such that $x^n \in B_r(x)$ for all $n \geq N_x$. Since $x^n \to y$, there exists some N_y such that $x^n \in B_r(y)$ for all $n \geq N_y$. But these statements are contradictory since $B_r(x) \cap B(y, r) = \emptyset$. We conclude that the successive terms of a convergent sequence cannot get arbitrarily close to two distinct points, so that the limit a convergent sequence is unique.

1.97 Let (x^n) be a sequence which converges to x. There exists some N such that

$$\rho(x^n - x) < 1$$

for all $n \geq N$. Let

$$R = \max\{\, \rho(x^1 - x), \rho(x^2 - x), \ldots, \rho(x^{N-1} - x), 1 \,\}$$

Then for all n, $\rho(x^n - x) \leq R$. That is every element x^n in the sequence (x^n) belongs to $B(x, R+1)$, the open ball about x of radius $R+1$. Therefore the sequence is bounded.

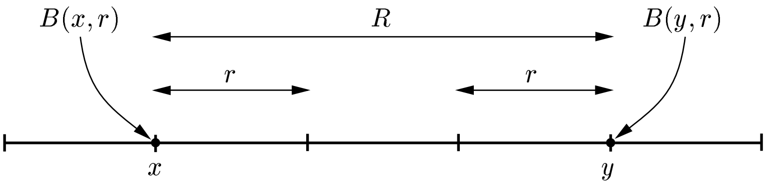

Figure 1.3: A convergent sequence cannot have two distinct limits

1.98 The share s^n of the nth guest is

$$s^n = \frac{1}{2}^n$$

$$\lim s^n = 0$$

However, $s^n > 0$ for all n. There is no limit to the number of guests who will get a share of the cake, although the shares will get vanishingly small for large parties.

1.99 Suppose $x^n \to x$. That is, there exists some N such that $\rho(x^n, x) < \epsilon/2$ for all $n \geq N$. Then, for all $m, n \geq N$

$$\begin{aligned} \rho(x^m, x^n) &\leq \rho(x^m, x) + \rho(x, x^n) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

1.100 Let (x^n) be a Cauchy sequence. There exists some N such that

$$\rho(x^n - x^N) < 1$$

for all $n \geq N$. Let

$$R = \max\{\,\rho(x^1 - x^N), \rho(x^2 - x^N), \dots, \rho(x^{N-1} - x^N), 1\,\}$$

Every x^n belongs to $B(x^N, R+1)$, the ball of radius $R+1$ centered on x^N.

1.101 Let (x^n) be a bounded increasing sequence in $\Re$ and let $S = \{\, x^n \,\}$ be the set of elements of (x^n). Let b be the least upper bound of S. We show that $x^n \to b$.

First observe that $x^n \leq b$ for every n (since b is an upper bound). Since b is the least upper bound, for every $\epsilon > 0$ there exists some element x^N such that $x^N > b - \epsilon$. Since (x^n) is increasing, we must have

$$b - \epsilon < x^n \leq b \text{ for every } \; n \geq N$$

That is, for every $\epsilon > 0$ there exists an N such that

$$\rho(x^n, x) < \epsilon \text{ for every } \; n \geq N$$

$x^n \to b$.

1.102 If $\beta > 1$, the sequence $\beta, \beta^2, \beta^3, \dots$ is unbounded.

Otherwise, if $\beta \leq 1$, $\beta^n \leq \beta^{n-1}$ and the sequence is decreasing and bounded by $\beta \leq 1$. Therefore the sequence converges (Exercise 1.101). Let $x = \lim_{n\to\infty}$. Then

$$\beta^{n+1} = \beta\beta^n$$

and therefore

$$x = \lim_{n\to\infty} \beta^{n+1} = \beta \lim_{n\to\infty} \beta^n = \beta x$$

which can be satisfied if and only if

- $\beta = 1$, in which case $x = \lim 1^n = 1$
- $x = 0$ when $0 \leq \beta < 1$

Therefore

$$\beta^n \to 0 \iff \beta < 1$$

1.103 1. For every $x \in \Re$

$$(x - \sqrt{2})^2 \geq 0$$

Expanding

$$\begin{aligned} x^2 - 2\sqrt{2}x + 2 &\geq 0 \\ x^2 + 2 &\geq 2\sqrt{2}x \end{aligned}$$

Dividing by x

$$x + \frac{2}{x} \geq 2\sqrt{2}$$

for every $x > 0$. Therefore

$$\frac{1}{2}\left(x + \frac{2}{x}\right) \geq \sqrt{2}$$

2. Let (x^n) be the sequence defined in Example 1.64. That is

$$x^n = \frac{1}{2}\left(x^{n-1} + \frac{2}{x^{n-1}}\right)$$

Starting from $x^0 = 2$, it is clear that $x^n \geq 0$ for all n. Substituting in

$$\frac{1}{2}\left(x + \frac{2}{x}\right) \geq \sqrt{2}$$

$$x^n = \frac{1}{2}\left(x^{n-1} + \frac{2}{x^{n-1}}\right) \geq \sqrt{2}$$

That is $x^n \geq \sqrt{2}$ for every n. Therefore for every n

$$\begin{aligned} x^n - x^{n+1} &= x^n - \frac{1}{2}\left(x^n + \frac{2}{x^n}\right) \\ &= \frac{1}{2}\left(x^n - \frac{2}{x^n}\right) \\ &\geq \frac{1}{2}\left(x^n - \frac{2}{\sqrt{2}}\right) \\ &= x^n - \sqrt{2} \\ &\geq 0 \end{aligned}$$

This implies that $x^{n+1} \leq x^n$. Consequently $\sqrt{2} \leq x^n \leq 2$ for every n. (x^n) is a bounded monotone sequence. By Exercise 1.101, $x^n \to x$. The limit x satisfies the equation

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right)$$

Solving, this implies $x^2 = 2$ or $x = \sqrt{2}$ as required.

1.104 The following sequence approximates the square root of any positive number a

$$x^1 = a$$
$$x^{n+1} = \frac{1}{2}(x^n + \frac{a}{x^n})$$

1.105 Let $x \in \overline{S}$. If $x \in S$, then x is the limit of the sequence $(x, x, x, \dots)$. If $x \notin S$, then x is a boundary point of S. For every n, the ball $B(x, 1/n)$ contains a point $x^n \in S$. From the sequence of open balls $B(x, 1/n)$ for $n = 1, 2, 3, \dots$, we can generate of a sequence of points x^n which converges to x.

Conversely, assume that x is the limit of a sequence (x^n) of points in S. Either $x \in S$ and therefore $x \in \overline{S}$. Or $x \notin S$. Since $x^n \to x$, every neighborhood of x contains points x^n of the sequence. Hence, x is a boundary point of S and $x \in \overline{S}$.

1.106 S is closed if and only if $S = \overline{S}$. The result follows from Exercise 1.105.

1.107 Let S be a closed subset of a complete metric space X. Let (x^n) be a Cauchy sequence in S. Since X is complete, $x^n \to x \in X$. Since S is closed, $x \in S$ (Exercise 1.106).

1.108 Since $d(S^n) \to 0$, S cannot contain more than one point. Therefore, it suffices to show that S is nonempty. Choose some x^n from each S^n. Since $d(S^n) \to 0$, (x^n) is a Cauchy sequence. Since X is complete, there exists some $x \in X$ such that $x^n \to x$.

Choose some m. Since the sets are nested, the subsequence $\{ x^n : n \geq m \} \subseteq S^m$. Since S^m is closed, $x \in S^m$ (Exercise 1.106). Since $x \in S^m$ for every m

$$x \in \bigcap_{m=1}^{\infty} S^m$$

1.109 If player 1 picks closed balls whose radius decreases by at least half after each pair of moves, then $\{ S^1, S^3, S^5, \dots \}$ is a nested sequence of closed sets which has a nonempty intersection (Exercise 1.108).

1.110 Let (x^n) be a sequence in $S \subseteq T$ with S closed and T compact. Since T is compact, there exists a convergent subsequence $x^m \to x \in T$. Since S is closed, we must have $x \in S$ (Exercise 1.106). Therefore (x^n) contains a subsequence which converges in S, so that S is compact.

1.111 Let (x^n) be a Cauchy sequence in a metric space. For every $\epsilon > 0$, there exists N such that

$$\rho(x^m, x^n) < \epsilon/2 \text{ for all } m, n \geq N$$

Trivially, if (x^n) converges, it has a convergent subsequence (the whole sequence).

Conversely, assume that (x^n) has a subsequence (x^m) which converges to x. That is, there exists some M such that

$$\rho(x^m, x) < \epsilon/2 \text{ for all } m \geq M$$

Therefore, by the triangle inequality

$$\rho(x^n, x) \leq \rho(x^n, x^M) + \rho(x^M, x) < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $n \geq \max M, N$

1.112 We proceed sequentially as follows. Choose any x_1 in X. If the open ball $B(x_1, r)$ contains X, we are done. Otherwise, choose some $x_2 \notin B(x_1, r)$ and consider the set $\bigcup_{i=1}^{2} B(x_i, r)$. If this set contains X, we are done. Otherwise, choose some $x_3 \notin \bigcup_{i=1}^{2} B(x_i, r)$ and consider $\bigcup_{i=1}^{3} B(x_i, r)$

The process must terminate with a finite number of open balls. Otherwise, if the process could be continued indefinitely, we could construct an infinite sequence $(x_1, x_2, x_3, \dots)$ which had no convergent subsequence. The would contradict the compactness of X.

1.113 Assume X is compact. The previous exercise showed that X is totally bounded. Further, since every sequence has a convergent subsequence, every Cauchy sequence converges (Exercise 1.111). Therefore X is complete.

Conversely, assume that X is complete and totally bounded and let $S_1 = \{ x_1^1, x_1^2, x_1^3, \dots \}$ be an infinite sequence of points in X. Since X is totally bounded, it is covered by a *finite* collection of open balls of radius $1/2$. S_1 has a subsequence $S_2 = \{ x_2^1, x_2^2, x_2^3, \dots \}$ all of whose points lie in one of the open balls. Similarly, S_2 has a subsequence $S_3 = \{ x_3^1, x_3^2, x_3^3, \dots \}$ all of whose points lie in an open ball of radius $1/3$. Continuing in this fashion, we construct a sequence of subsequences, each of which lies in a ball of smaller and smaller radius. Consequently, successive terms of the "diagonal" subsequence $\{ x_1^1, x_2^2, x_3^3, \dots \}$ get closer and closer together. That is, S is a Cauchy sequence. Since X is complete, S converges in X and S_1 has a convergent subsequence S. Hence, X is compact.

1.114

1. Every big set $T \in \mathcal{B}$ has a least two distinct points. Hence $d(T) > 0$ for every $T \in \mathcal{B}$.
2. Otherwise, there exists n such that $d(T) \geq 1/n$ for every $T \in \mathcal{B}$ and therefore $\delta = \inf_{T \in \mathcal{B}} d(T) \geq 1/n > 0$.
3. Choose a point x^n in each T_n. Since X is compact, the sequence (x^n) has a convergent subsequence (x^m) which converges to some point $x_0 \in X$.
4. The point x_0 belongs to at least one S_0 in the open cover $\mathcal{C}$. Since S_0 is open, there exists some open ball $B_r(x_0) \subseteq S_0$.
5. Consider the concentric ball $B_{r/2}(x_0)$. Since (x^m) is a convergent subsequence, there exists some M such that $x^m \in B_{r/2}(x)$ for every $m \geq M$.
6. Choose some $n_0 \geq \min\{ M, 2/r \}$. Then $1/n_0 < r/2$ and $d(T_{n_0}) < 1/n_0 < r/2$. $x_{n_0} \in T_{n_0} \cap B_{r/2}(x)$ and therefore (Exercise 1.90) $T_{n_0} \subseteq B_r(x) \subseteq S^0$.

This contradicts the assumption that T_n is a big set. Therefore, we conclude that $\delta > 0$.

1.115

1. X is totally bounded (Exercise 1.112). Therefore, for every $r > 0$, there exists a finite number of open balls $B_r(x_n)$ such that
$$X = \bigcup_{i=1}^{n} B_r(x_i)$$
2. $d(B_r(x_i)) = 2r < \delta$. By definition of the Lebesgue number, every $B_r(x_i)$ is contained in some $S_i \in \mathcal{C}$.
3. The collection of open balls $\{B_r(x_i)\}$ covers X. Therefore, fore every $x \in X$, there exists i such that
$$x \in B_r(x_i) \subseteq S_i$$

Therefore, the finite collection $S_1, S_2, \dots, S_n$ covers X.

1.116 For any family of subsets $\mathcal{C}$

$$\bigcap_{S \in \mathcal{C}} S = \emptyset \iff \bigcup_{S \in \mathcal{C}} S^c = X$$

Suppose to the contrary that $\mathcal{C}$ is a collection of closed sets with the finite intersection property, but that $\bigcap_{S \in \mathcal{C}} S = \emptyset$. Then $\{ S^c : S \in \mathcal{C} \}$ is a open cover of X which does not have a finite subcover. Consequently X cannot be compact.

Conversely, assume every collection of closed sets with the finite intersection property has a nonempty intersection. Let $\mathcal{B}$ be an open cover of X. Let

$$\mathcal{C} = \{ S \subseteq X : S^c \in \mathcal{B} \}$$

That is

$$\bigcup_{S \in \mathcal{C}} S^c = X \text{ which implies } \bigcap_{S \in \mathcal{C}} S = \emptyset$$

Consequently, $\mathcal{C}$ does not have the finite intersection property. There exists a finite subcollection $\{ S_1, S_2, \dots, S_n \}$ such that

$$\bigcap_{i=1}^{n} S_i = \emptyset$$

which implies that

$$\bigcup_{i=1}^{n} S_i^c = X$$

$\{ S_1^c, S_2^c, \dots, S_n^c \}$ is a finite subcover of X. Thus, X is compact.

1.117 Every finite collection of nested (nonempty) sets has the finite intersection property. By Exercise 1.116, the sequence has a non-empty intersection. (Note: every set S_i is a subset of the compact set S_1.)

1.118 **(1)** $\implies$ **(2)** Exercises 1.114 and 1.115.

(2) $\implies$ **(3)** Exercise 1.116

(3) $\implies$ **(1)** Let X be a metric space in which every collection of closed subsets with the finite intersection property has a finite intersection. Let (x^n) be a sequence in X. For any n, let S_n be the tail of the sequence minus the first n terms, that is

$$S_n = \{ x^m : m = n+1, n+2, \dots \}$$

The collection $(\overline{S_n})$ has the finite intersection property since, for any finite set of integers $\{ n_1, n_2, \dots, n_k \}$

$$\bigcap_{j=1}^{k} \overline{S_{n_j}} \subseteq \overline{S_K} \neq \emptyset$$

where $K = \max\{ n_1, n_2, \dots, n_k \}$. Therefore

$$\bigcap_{n=1}^{\infty} \overline{S_n} \neq \emptyset$$

Choose any $x \in \bigcup_{n=1}^{\infty} \overline{S_n}$. That is, $x \in \overline{S_n}$ for each $n = 1, 2, \dots$. Thus, for every $r > 0$ and $n = 1, 2, \dots$, there exists some $x_n \in B_r(x) \cap S_n$

We construct a subsequence as follows. For $k = 1, 2, \dots$, let x^k be the first term in S_k which belongs to $B_{1/k}(x)$. Then, (x^k) is a subsequence of (x^n) which converges to x. We conclude that every sequence has a convergent subsequence.

1.119 Assume (x^n) is a bounded sequence in $\Re$. Without loss of generality, we can assume that $\{ x^n \} \subset [0, 1]$. Divide $I^0 = [0, 1]$ into two sub-intervals $[0, 1/2]$ and $[1/2, 1]$. At least one of the sub-intervals must contain an infinite number of terms of the sequence. Call this interval I^1. Continuing this process of subdivision, we obtain a nested sequence of intervals

$$I^0 \supset I^1 \supset I^2 \supset \dots$$

each of which contains an infinite number of terms of the sequence. Consequently, we can construct a subsequence (x^m) with $x^m \in I^m$. Furthermore, the intervals get smaller and smaller with $d(I^n) \to 0$, so that (x^m) is a Cauchy sequence. Since $\Re$ is complete, the subsequence (x^m) converges to $x \in \Re$.

Note how we implicitly called on the Axiom of Choice (Remark 1.5) in choosing a subsequence from the nested sequence of intervals.

1.120 Let (x^n) be a Cauchy sequence in $\Re$. That is, for every $\epsilon > 0$, there exists N such that $|x^n - x^m| < \epsilon$ for all $m, n \geq N$. (x^n) is bounded (Exercise 1.100) and hence by the Bolzano-Weierstrass theorem, it has a convergent subsequence (x^m) with $x^m \to x \in \Re$. Choose x^r from the convergent subsequence such that $r \geq N$ and $|x^r - x| < \epsilon/2$. By the triangle inequality

$$|x^n - x| \leq |x^n - x^r| + |x^r - x| < \epsilon/2 + \epsilon/2 = \epsilon$$

Hence the sequence (x^n) converges to $x \in \Re$.

1.121 Since X_1 and X_2 are linear spaces, $\mathbf{x}_1 + \mathbf{y}_1 \in X_1$ and $\mathbf{x}_2 + \mathbf{y}_2 \in X_2$, so that $(\mathbf{x}_1 + \mathbf{y}_1, \mathbf{x}_2 + \mathbf{y}_2) \in X_1 \times X_2$. Similarly $(\alpha\mathbf{x}_1, \alpha\mathbf{x}_2) \in X_1 \times X_2$ for every $(\mathbf{x}_1, \mathbf{x}_2) \in X_1 \times X_2$. Hence, $X = X_1 \times X_2$ is closed under addition and scalar multiplication.

With addition and scalar multiplication defined component-wise, X inherits the arithmetic properties (like associativity) of its constituent spaces. Verifying this would proceed identically as for $\Re^n$. It is straightforward though tedious. The zero element in X is $\mathbf{0} = (0_1, 0_2)$ where 0_1 is the zero element in X_1 and 0_2 is the zero element in X_2. Similarly, the inverse of $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ is $-\mathbf{x} = (-\mathbf{x}_1, -\mathbf{x}_2)$.

1.122 1.

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= \mathbf{x} + \mathbf{z} \\
-\mathbf{x} + (\mathbf{x} + \mathbf{y}) &= -\mathbf{x} + (\mathbf{x} + \mathbf{z}) \\
(-\mathbf{x} + \mathbf{x}) + \mathbf{y} &= (-\mathbf{x} + \mathbf{x}) + \mathbf{z} \\
\mathbf{0} + \mathbf{y} &= \mathbf{0} + \mathbf{z} \\
\mathbf{y} &= \mathbf{z}
\end{aligned}$$

2.

$$\begin{aligned}\alpha\mathbf{x} &= \alpha\mathbf{y}\\ \frac{1}{\alpha}(\alpha\mathbf{x}) &= \frac{1}{\alpha}(\alpha\mathbf{y})\\ \left(\frac{1}{\alpha}\alpha\right)\mathbf{x} &= \left(\frac{1}{\alpha}\alpha\right)\mathbf{y}\\ \mathbf{x} &= \mathbf{y}\end{aligned}$$

3. $\alpha\mathbf{x} = \beta\mathbf{x}$ implies

$$(\alpha-\beta)\mathbf{x} = \alpha\mathbf{x} - \beta\mathbf{x} = \mathbf{0}$$

Provided $\mathbf{x} = \mathbf{0}$, we must have

$$(\alpha-\beta)\mathbf{x} = 0\mathbf{x}$$

That is $\alpha - \beta = 0$ which implies $\alpha = \beta$.

4.

$$\begin{aligned}(\alpha-\beta)\mathbf{x} &= (\alpha+(-\beta))\mathbf{x}\\ &= \alpha\mathbf{x} + (-\beta)\mathbf{x}\\ &= \alpha\mathbf{x} - \beta\mathbf{x}\end{aligned}$$

5.

$$\begin{aligned}\alpha(\mathbf{x}-\mathbf{y}) &= \alpha(\mathbf{x}+(-1)\mathbf{y})\\ &= \alpha\mathbf{x} + \alpha(-1)\mathbf{y}\\ &= \alpha\mathbf{x} - \alpha\mathbf{y}\end{aligned}$$

6.

$$\begin{aligned}\alpha\mathbf{0} &= \alpha(\mathbf{x}+(-\mathbf{x}))\\ &= \alpha\mathbf{x} + \alpha(-\mathbf{x})\\ &= \alpha\mathbf{x} - \alpha\mathbf{x}\\ &= \mathbf{0}\end{aligned}$$

1.123 The linear hull of the vectors $\{(1,0),(0,2)\}$ is

$$\begin{aligned}\text{lin}\ \{(1,0),(0,2)\} &= \left\{\alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\2\end{pmatrix}\right\}\\ &= \left\{\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}\right\}\\ &= \Re^2\end{aligned}$$

The linear hull of the vectors $\{(1,0),(0,2)\}$ is the whole plane $\Re^2$. Figure 1.4 illustrates how any vector in $\Re^2$ can be obtained as a linear combination of $\{(1,0),(0,2)\}$.

1.124 1. From the definition of α,

$$\alpha_S = w(S) - \sum_{T \subsetneq S} \alpha_T$$

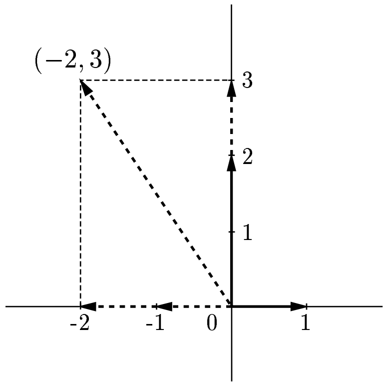

Figure 1.4: Illustrating the span of $\{(1,0),(0,2)\}$.

for every $S \subseteq N$. Rearranging

$$\begin{aligned} w(S) &= \alpha_S + \sum_{T \subsetneq S} \alpha_T \\ &= \sum_{T=S} \alpha_T + \sum_{T \subsetneq S} \alpha_T \\ &= \sum_{T \subseteq S} \alpha_T \end{aligned}$$

2.

$$\begin{aligned} \sum_{T \subseteq N} \alpha_T w_T(S) &= \sum_{T \subseteq S} \alpha_T w_T(S) + \sum_{T \not\subseteq S} \alpha_T w_T(S) \\ &= \sum_{T \subseteq S} \alpha_T 1 + \sum_{T \not\subseteq S} \alpha_T 0 \\ &= \sum_{T \subseteq S} \alpha_T 1 \\ &= w(S) \end{aligned}$$

1.125 1. Choose any $\mathbf{x} \in S$. By homogeneity $0\mathbf{x} = \theta \in S$.

2. For every $\mathbf{x} \in S$, $-\mathbf{x} = (-1)\mathbf{x} \in S$ by homogeneity.

1.126 Examples of subspaces in $\Re^n$ include:

1. The set containing just the null vector $\{\mathbf{0}\}$ is subspace.
2. Let $\mathbf{x}$ be any element in $\Re^n$ and let T be the set of all scalar multiples of $\mathbf{x}$

$$T = \{\alpha\mathbf{x} : \alpha \in \Re\}$$

   T is a line through the origin in $\Re^n$ and is a subspace.

3. Let S be the set of all n-tuples with zero first coordinate, that is

$$S = \{(x_1, x_2, \dots, x_n) : x_1 = 0,\ x_j \in \Re,\ j \neq 1\}$$

   For any $\mathbf{x}, \mathbf{y} \in S$

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (0, x_2, x_3, \dots, x_n) + (0, y_2, y_3, \dots, y_n) \\ &= (0, x_2 + y_2, x_3 + y_3, \dots, x_n + y_n) \in S \end{aligned}$$

Similarly

$$\begin{aligned}\alpha\mathbf{x} &= \alpha(0, x_2, x_3, \ldots, x_n) \\ &= (0, \alpha x_2, \alpha x_3, \ldots, \alpha x_n) \in S\end{aligned}$$

Therefore S is a subspace of $\Re^n$. Generalizing, any set of vectors with one or more coordinates identically zero is a subspace of $\Re^n$.

4. We will meet some more complicated subspaces in Chapter 2.

1.127 No, $-\mathbf{x} \notin \Re^n_+$ if $\mathbf{x} \in \Re^n_+$ unless $\mathbf{x} = 0$. $\Re^n_+$ is an example of a cone (Section 1.4.5).

1.128 **lin S is a subspace** Let $\mathbf{x}, \mathbf{y}$ be two elements in lin S. $\mathbf{x}$ is a linear combination of elements of S, that is

$$\mathbf{x} = \alpha_1 x_1 + \alpha_2 x_2 + \ldots \alpha_n x_n$$

Similarly

$$\mathbf{y} = \beta_1 x_1 + \beta_2 x_2 + \ldots \beta_n x_n$$

and

$$\mathbf{x} + \mathbf{y} = (\alpha_1 + \beta_1)x_1 + (\alpha_2 + \beta_2)x_2 + \cdots + (\alpha_n + \beta_n)x_n \in \text{lin } S$$

and

$$\alpha\mathbf{x} = \alpha\alpha_1 x_1 + \alpha\alpha_2 x_2 + \cdots + \alpha\alpha_n x_n \in \text{lin } S$$

This shows that lin S is closed under addition and scalar multiplication and hence is a subspace.

lin S is the smallest subspace containing S Let T be any subspace containing S. Then T contains all linear combinations of elements in S, so that lin $S \subset T$. Hence lin S is the smallest subspace containing S.

1.129 The previous exercise showed that lin S is a subspace. Therefore, if $S = \text{lin } S$, S is a subspace.

Conversely, assume that S is a subspace. Then S is the smallest subspace containing S, and therefore $S = \text{lin } S$ (again by the previous exercise).

1.130 Let $\mathbf{x}, \mathbf{y} \in S = S_1 \cap S_2$. Hence $\mathbf{x}, \mathbf{y} \in S_1$ and for any $\alpha, \beta \in \Re$, $\alpha\mathbf{x} + \beta\mathbf{y} \in S_1$. Similarly $\alpha\mathbf{x} + \beta\mathbf{y} \in S_2$ and therefore $\alpha\mathbf{x} + \beta\mathbf{y} \in S$. S is a subspace.

1.131 Let $S = S_1 + S_2$. First note that $0 = 0 + 0 \in S$. Suppose $\mathbf{x}, \mathbf{y}$ belong to S. Then there exist $\mathbf{s}_1, \mathbf{t}_1 \in S_1$ and $\mathbf{s}_2, \mathbf{t}_2 \in S_2$ such that $\mathbf{x} = \mathbf{s}_1 + \mathbf{s}_2$ and $\mathbf{y} = \mathbf{t}_1 + \mathbf{t}_2$. For any $\alpha, \beta \in \Re$,

$$\begin{aligned}\alpha\mathbf{x} + \beta\mathbf{y} &= \alpha(\mathbf{s}_1 + \mathbf{s}_2) + \beta(\mathbf{t}_1 + \mathbf{t}_2) \\ &= (\alpha\mathbf{s}_1 + \beta\mathbf{t}_1) + (\alpha\mathbf{s}_2 + \beta\mathbf{t}_2) \in S\end{aligned}$$

since $\alpha\mathbf{s}_1 + \beta\mathbf{t}_1 \in S_1$ and $\alpha\mathbf{s}_2 + \beta\mathbf{t}_2 \in S_2$.

1.132 Let

$$\begin{aligned}S_1 &= \{\, \alpha(1,0) : \alpha \in \Re \,\} \\ S_2 &= \{\, \alpha(0,1) : \alpha \in \Re \,\}\end{aligned}$$

S_1 and S_2 are respectively the horizontal and vertical axes in $\Re^2$. Their union is not a subspace, since for example

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin S_1 \cup S_2$$

However, any vector in $\Re^2$ can be written as the sum of an element of S_1 and an element of S_2. Therefore, their sum is the whole space $\Re^2$, that is

$$S_1 + S_2 = \Re^2$$

1.133 Assume that S is linearly dependent, that is there exists $\mathbf{x}_1, \dots, \mathbf{x}_n \in S$ and $\alpha_2, \dots, \alpha_n \in R$ such that

$$\mathbf{x}_1 = \alpha_2 \mathbf{x}_2 + \alpha_3 \mathbf{x}_3 + \dots, \alpha_n \mathbf{x}_n$$

Rearranging, this implies

$$1\mathbf{x}_1 - \alpha_2 \mathbf{x}_2 - \alpha_3 \mathbf{x}_3 - \dots \alpha_n \mathbf{x}_n = 0$$

Conversely, assume there exist $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbf{x}$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \Re$ such that

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 \dots + \alpha_n \mathbf{x}_n = 0$$

Assume without loss of generality that $\alpha_1 \neq 0$. Then

$$\mathbf{x}_1 = -\frac{\alpha_2}{\alpha_1}\mathbf{x}_2 - \frac{\alpha_3}{\alpha_1}\mathbf{x}_3 - \dots - \frac{\alpha_n}{\alpha_1}\mathbf{x}_n$$

which shows that

$$\mathbf{x}_1 \in \text{lin } S \setminus \{\mathbf{x}_1\}$$

1.134 Assume $\{(1,1,1), (0,1,1), (0,0,1)\}$ are linearly dependent. Then there exists $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or equivalently

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_1 + \alpha_2 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \end{aligned}$$

which imply that

$$\alpha_1 = \alpha_2 = \alpha_3 = 0$$

Therefore $\{(1,1,1), (0,1,1), (0,0,1)\}$ are linearly independent.

1.135 Suppose on the contrary that U is linearly dependent. That is, there exists a set of games $\{u_{T_1}, u_{T_2}, \dots, u_{T_m}\}$ and nonzero coefficients $(\alpha_1, \alpha_2, \dots, \alpha_m)$ such that (Exercise 1.133)

$$\alpha_1 u_{T_1} + \alpha_2 u_{T_2} + \dots + \alpha_m u_{T_m} = \mathbf{0} \tag{1.2}$$

Assume that the coalitions are ordered so that T_1 has the smallest number of players of any of the coalitions $T_1, T_2, \ldots, T_m$. This implies that no coalition $T_2, T_3, \ldots, T_m$ is a subset of T_1 and

$$u_{T_j}(T_1) = 0 \qquad \text{for every } \; j = 2, 3, \ldots, n \tag{1.3}$$

Using (1.2), u_{T_1} can be expressed as a linear combination of the other games,

$$u_{T_1} = -1/\alpha_1 \sum_{j=2}^{m} \alpha_j u_{T_j} \tag{1.4}$$

Substituting (1.3) this implies that

$$u_{T_1}(T_1) = 0$$

whereas

$$u_T(T) = 1 \qquad \text{for every } \; T$$

by definition. This contradiction establishes that the set U is linearly independent.

1.136 If S is a subspace, then $\mathbf{0} \in S$ and

$$\alpha \mathbf{x}_1 = \mathbf{0}$$

with $\alpha \neq 0$ and $\mathbf{x}_1 = \mathbf{0}$ (Exercise 1.122). Therefore S is linearly dependent (Exercise 1.133).

1.137 Suppose $\mathbf{x}$ has two representations, that is

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_n \mathbf{x}_n$$
$$\mathbf{x} = \beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \ldots + \beta_n \mathbf{x}_n$$

Subtracting

$$\mathbf{0} = (\alpha_1 - \beta_1)\mathbf{x}_1 + (\alpha_2 - \beta_2)\mathbf{x}_2 + \ldots + (\alpha_n - \beta_n)\mathbf{x}_n \tag{1.5}$$

Since $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, , \mathbf{x}_n\}$ is linearly independent, (1.5) implies that $\alpha_i = \beta_i = 0$ for all i (Exercise 1.133)

1.138 Let P be the set of all linearly independent subsets of a linear space X. P is partially ordered by inclusion. Every chain $C = \{S_\alpha\} \subseteq P$ has an upper bound, namely $\bigcup_{S \in C} S$. By Zorn's lemma, P has a maximal element B. We show that B is a basis for X.

B is linearly independent since $B \in P$. Suppose that B does not span X so that $\text{lin}\, B \subset X$. Then there exists some $\mathbf{x} \in X \setminus \text{lin}\, B$. The set $B \cup \{\mathbf{x}\}$ is a linearly independent and contains B, which contradicts the assumption that B is the maximal element of P. Consequently, we conclude that B spans X and hence is a basis.

1.139 Exercise 1.134 established that the set $B = \{\,(1,1,1), (0,1,1), (0,0,1)\}$ is linearly independent. Since $\dim R^3 = 3$, any other vectors must be linearly dependent on B. That is $\text{lin}\, B = \Re^3$. B is a basis.

By a similar argument to exercise 1.134, it is readily seen that $\{(1,0,0), (0,1,0), (0,0,1)\}$ is linearly independent and hence constitutes a basis.

1.140 Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ and $B = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m\}$ be two bases for a linear space X. Let

$$S_1 = \{b_1\} \cup A = \{\mathbf{b}_1, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$$

S is linearly dependent (since $b_1 \in \text{lin } A$) and spans X. Therefore, there exists $\alpha_1, \alpha_2, \dots, \alpha_n$ and β_1 such that

$$\beta_1 \mathbf{b}_1 + \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \dots + \alpha_n \mathbf{a}_n = \mathbf{0}$$

At least one $\alpha_i \neq 0$. Deleting the corresponding element $\mathbf{a}_i$, we obtain another set S_1' of n elements

$$S_1' = \{\mathbf{b}_1, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_n\}$$

which is also spans X. Adding the second element from B, we obtain the $n+1$ element set

$$S_2 = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_n\}$$

which again is linearly dependent and spans X.

Continuing in this way, we can replace m vectors in A with the m vectors from B while maintaining a spanning set. This process cannot eliminate all the vectors in A, because this would imply that B was linearly dependent. (Otherwise, the remaining $\mathbf{b}_i$ would be linear combinations of preceding elements of B.) We conclude that necessarily $m \leq n$. Reversing the process and replacing elements of B with elements of A establishes that $n \leq m$. Together these inequalities imply that $n = m$ and A and B have the same number of elements.

1.141 Suppose that the coalitions are ordered in some way, so that

$$\mathcal{P}(N) = \{S_0, S_1, S_2, \dots, S_{2^n-1}\}$$

with $S_0 = \emptyset$. There are 2^n coalitions. Each game $G \in \mathcal{G}^N$ corresponds to a unique list of length 2^n of coalitional worths

$$\mathbf{v} = (v_0, v_1, v_2, \dots, v_{2^n-1})$$

with $v_0 = 0$. That is, each game defines a vector $v = (0, v_1, \dots, v_{2^n-1}) \in \Re^{2^n}$ and conversely each vector $v \in \Re^{2^n}$ (with $v_0 = 0$) defines a game. Therefore, the space of all games $\mathcal{G}^N$ is formally identical to the subspace of $\Re^{2^n}$ in which the first component is identically zero, which in turn is equivalent to the space $\Re^{2^n-1}$. Thus, $\mathcal{G}^N$ is a $2^n - 1$-dimensional linear space.

1.142 For illustrative purposes, we present two proofs, depending upon whether the linear space is assumed to be finite dimensional or not. In the finite dimensional case, a constructive proof is possible, which forms the basis for practical algorithms for constructing a basis.

Let S be a linearly independent set in a linear space X.

X **is finite dimensional** Let $n = \dim X$. Assume S has m elements and denote it S_m.

If $\text{lin } S_m = X$, then S_m is a basis and we are done. Otherwise, there exists some $\mathbf{x}_{m+1} \in X \setminus \text{lin } S_m$. Adding $\mathbf{x}_{m+1}$ to S_m gives a new set of $m+1$ elements

$$S_{m+1} = S_m \cup \{\mathbf{x}_{m+1}\}$$

which is also linearly independent (since $\mathbf{x}_{m+1} \notin \text{lin } S_m$).

If lin $S_{m+1} = X$, then S_{m+1} is a basis and we are done. Otherwise, there exists some $\mathbf{x}_{m+2} \in X \setminus \text{lin } S_{m+1}$. Adding $\mathbf{x}_{m+2}$ to S_{m+1} gives a new set of $m+2$ elements

$$S_{m+2} = S_{m+1} \cup \{\, \mathbf{x}_{m+2} \,\}$$

which is also linearly independent (since $\mathbf{x}_{m+2} \notin \text{lin } S_{m+2}$).

Repeating this process, we can construct a sequence of linearly independent sets $S_m, S_{m+1}, S_{m+2} \ldots$ such that lin $S_m \subsetneqq \text{lin } S_{m+1} \subsetneqq \text{lin } S_{m+2} \cdots \subseteq X$. Eventually, we will reach a set which spans X and hence is a basis.

X **is possibly infinite dimensional** For the general case, we can adapt the proof of the existence of a basis (Exercise 1.138), restricting P to be the class of all linearly independent subsets of X *containing* S.

1.143 Otherwise (if a set of $n+1$ elements was linearly independent), it could be extended to basis at least $n+1$ elements (exercise 1.142). This would contradict the fundamental result that all bases have the same number of elements (Exercise 1.140).

1.144 Every basis is linearly independent. Conversely, let $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\,\}$ be a set of linearly independent elements in an n-dimensional linear space X. We have to show that lin $B = X$.

Take any $\mathbf{x} \in X$. The set

$$B \cup \{\mathbf{x}\} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \mathbf{x}\,\}$$

must be linearly dependent (Exercise 1.143). That is there exists numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$, not all zero, such that

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n + \alpha\mathbf{x} = 0 \tag{1.6}$$

Furthermore, it must be the case that $\alpha \neq 0$ since otherwise

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n = 0$$

which contradicts the linear independence of A. Solving (1.6) for $\mathbf{x}$, we obtain

$$\mathbf{x} = \frac{1}{\alpha}\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ \cdots + \alpha_n\mathbf{x}_n$$

Since $\mathbf{x}$ was an arbitrary element of X, we conclude that B spans X and hence B is a basis.

1.145 A basis spans X. To establish the converse, assume that $B = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\,\}$ is a set of n elements which span X. If S is linearly dependent, then one element is linearly dependent on the other elements. Without loss of generality, assume that $\mathbf{x}_1 \in \text{lin } B \setminus \{\mathbf{x}_1\}$. Deleting $\mathbf{x}_1$ the set

$$B \setminus \{\mathbf{x}_1\} = \{\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n\}$$

also spans X. Continuing in this fashion by eliminating dependent elements, we finish with a linearly independent set of $m < n$ elements which spans X. That is, we can find a basis of $m < n$ elements, which contradicts the assumption that the dimension of X is n (Exercise 1.140). Thus any set of n vectors which spans X must be linearly independent and hence a basis.

1.146 We have previously shown

- that the set U is linearly independent (Exercise 1.135).
- the space $\mathcal{G}^N$ has dimension 2^{n-1} (Exercise 1.141).

There are 2^{n-1} distinct T-unanimity games u_T in U. Hence U spans the 2^{n-1} space $\mathcal{G}^N$. Alternatively, note that any game $w \in \mathcal{G}^N$ can be written as a linear combination of T-unanimity games (Exercise 1.75).

1.147 Let $B = \{\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^m\}$ be a basis for S. Since B is linearly independent, $m \leq n$ (Exercise 1.143). There are two possibilities.

Case 1: $m = n$**.** B is a set of n linearly independent elements in an n-dimensional space X. Hence B is a basis for X and $S = \text{lin } B = X$.

Case 2: $m < n$**.** Since B is linearly independent but cannot be a basis for the n-dimensional space X, we must have $S = \text{lin } B \subset X$.

Therefore, we conclude that if $S \subset X$ is a proper subspace, it has a lower dimension than X.

1.148 Let $\alpha_1, \alpha_2, \alpha_3$ be the coordinates of $(1,1,1)$ for the basis $\{(1,1,1),(0,1,1),(0,0,1)\}$. That is

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}0\\0\\1\end{pmatrix}$$

which implies that $\alpha_1 = 1, \alpha_2 = \alpha_3 = 0$. Therefore $(1,0,0)$ are the required coordinates of the $(1,1,1)$ with respect to the basis $\{(1,1,1),(0,1,1),(0,0,1)\}$.

$(1,1,1)$ are the coordinates of the vector $(1,1,1)$ with respect to the standard basis.

1.149 A subset S of a linear space X is a subspace of X if

$$\alpha\mathbf{x} + \beta\mathbf{y} \in S \text{ for every } \mathbf{x}, \mathbf{y} \in S \text{ and } \text{ for every } \alpha, \beta \in \Re$$

Letting $\beta = 1 - \alpha$, this implies that

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S \qquad \text{for every } \mathbf{x}, \mathbf{y} \in S \text{ and } \alpha \in \Re$$

S is an affine set.

Conversely, suppose that S is an affine set containing $\mathbf{0}$, that is

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S \qquad \text{for every } \mathbf{x}, \mathbf{y} \in S \text{ and } \alpha \in \Re$$

Letting $\mathbf{y} = \mathbf{0}$, this implies that

$$\alpha\mathbf{x} \in S \qquad \text{for every } \mathbf{x} \in S \text{ and } \alpha \in \Re$$

so that S is homogeneous. Now letting $\alpha = \frac{1}{2}$, for every $\mathbf{x}$ and $\mathbf{y}$ in S,

$$\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} \in S$$

and homogeneity implies

$$\mathbf{x} + \mathbf{y} = 2\left(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}\right) \in S$$

S is also additive. Hence S is subspace.

1.150 For any $\mathbf{x} \in S$, let

$$V = S - \mathbf{x} = \{ \mathbf{v} \in X : \mathbf{v} + \mathbf{x} \in S \}$$

V is an affine set For any $\mathbf{v}^1, \mathbf{v}^2 \in V$, there exist corresponding $\mathbf{s}^1, \mathbf{s}^2 \in S$ such that $\mathbf{v}^1 = \mathbf{s}^1 - \mathbf{x}$ and $\mathbf{v}^2 = \mathbf{s}^2 - \mathbf{x}$ and therefore

$$\begin{aligned} \alpha \mathbf{v}^1 + (1-\alpha)\mathbf{v}^2 &= \alpha(\mathbf{s}^1 - \mathbf{x}) + (1-\alpha)(\mathbf{s}^1 - \mathbf{x}) \\ &= \alpha \mathbf{s}^1 + (1-\alpha)\mathbf{s}^2 - \alpha \mathbf{x} + (1-\alpha)\mathbf{x} \\ &= \mathbf{s} - \mathbf{x} \end{aligned}$$

where $\mathbf{s} = \alpha s^1 + (1-\alpha)s^2 \in S$. There V is an affine set.

V is a subspace Since $\mathbf{x} \in S$, $\mathbf{0} = \mathbf{x} - \mathbf{x} \in V$. Therefore V is a subspace (Exercise 1.149).

V is unique Suppose that there are two subspaces V^1 and V^2 such that $S = V^1 + \mathbf{x}^1$ and $S = V^2 + \mathbf{x}^2$. Then

$$\begin{aligned} V_1 + \mathbf{x}_1 &= V_2 + \mathbf{x}_2 \\ V_1 &= V_2 + (\mathbf{x}_2 - \mathbf{x}_1) \\ &= V_2 + \mathbf{x} \end{aligned}$$

where $\mathbf{x} = \mathbf{x}_2 - \mathbf{x}_1 \in X$. Therefore V_1 is parallel to V_2.

Since V_1 is a subspace, $\mathbf{0} \in V_1$ which implies that $-\mathbf{x} \in V_2$. Since V_2 is a subspace, this implies that $\mathbf{x} \in V_2$ and $V_2 + \mathbf{x} \subseteq V_2$. Therefore $V_1 = V_2 + \mathbf{x} \subseteq V_2$. Similarly, $V_2 \subseteq V_1$ and hence $V_1 = V_2$. Therefore the subspace V is unique.

1.151 Let $S \parallel T$ denote the relation S is parallel to T, that is

$$S \parallel T \iff S = T + \mathbf{x} \text{ for some } \mathbf{x} \in X$$

The relation $\parallel$ is

reflexive $S \parallel S$ since $S = S + \mathbf{0}$

transitive Assume $S = T + \mathbf{x}$ and $T = U + \mathbf{y}$. Then $S = U + (\mathbf{x} + \mathbf{y})$

symmetric $S = T + \mathbf{x} \implies T = S + (-\mathbf{x})$

Therefore $\parallel$ is an equivalence relation.

1.152 See exercises 1.130 and 1.162.

1.153

1. Exercise 1.150
2. Assume $\mathbf{x}_0 \in V$. For every $\mathbf{x} \in H$

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v} = \mathbf{w} \in V$$

which implies that $H \subseteq V$. Conversely, assume $H = V$. Then $\mathbf{x}_0 = \mathbf{0} \in V$ since V is a subspace.

3. By definition, $H \subset X$. Therefore $V = H - \mathbf{x} \subset X$.
4. Let $\mathbf{x}_1 \notin V$. Suppose to the contrary

$$\text{lin}\,\{\mathbf{x}_1, V\} = V' \subset X$$

Then

$$H' = \mathbf{x}_0 + V'$$

is an affine set (Exercise 1.150) which strictly contains H. This contradicts the definition of H as a maximal proper affine set.

5. Let $\mathbf{x}_1 \notin V$. By the previous part, $\mathbf{x} \in \text{lin}\ \{\mathbf{x}_1, V\}$. That is, there exists $\alpha \in \Re$ such that

$$\mathbf{x} = \alpha \mathbf{x}_1 + \mathbf{v} \text{ for some } \mathbf{v} \in V$$

To see that α is unique, suppose that there exists $\beta \in \Re$ such that

$$\mathbf{x} = \beta \mathbf{x}_1 + \mathbf{v}' \text{ for some } \mathbf{v}' \in V$$

Subtracting

$$\mathbf{0} = (\alpha - \beta)\mathbf{x}_1 + (\mathbf{v} - \mathbf{v}')$$

which implies that $\alpha = \beta$ since $\mathbf{x}_1 \notin V$.

1.154 Assume $\mathbf{x}, \mathbf{y} \in X$. That is, $\mathbf{x}, \mathbf{y} \in \Re^n$ and

$$\sum_{i \in N} x_i = \sum_{i \in N} y_i = w(N)$$

For any $\alpha \in \Re$, $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \Re^n$ and

$$\begin{aligned}\sum_{i \in N} \alpha x_i + (1-\alpha) y_i &= \alpha \sum_{i \in N} x_i + (1-\alpha) \sum_{i \in N} y_i \\ &= \alpha w(N) + (1-\alpha) w(N) \\ &= w(N)\end{aligned}$$

Hence X is an affine subset of $\Re^n$.

1.155 See Exercise 1.129.

1.156 No. A straight line through any two points in $\Re^n_+$ extends outside $\Re^n_+$. Put differently, the affine hull of $\Re^n_+$ is the whole space $\Re^n$.

1.157 Let

$$\begin{aligned}V &= \text{aff}\ S - \mathbf{x}_1 \\ &= \text{aff}\ \{\mathbf{0}, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1\}\end{aligned}$$

V is a subspace ($\mathbf{0} \in V$) and

$$\text{aff}\ S = V + \mathbf{x}_1$$

and

$$\dim \text{aff}\ S = \dim V$$

Note that the choice of $\mathbf{x}_1$ is arbitrary.

S is affinely dependent if and only if there exists some $\mathbf{x}_k \in S$ such that $\mathbf{x}_k \in \text{aff}\ (S \setminus \{\mathbf{x}_k\})$. Since the choice of $\mathbf{x}_1$ is arbitrary, we assume that $\mathbf{x}_k \neq \mathbf{x}_1$.

$$\begin{aligned}\mathbf{x}_k \in \text{aff}\ (S \setminus \{\mathbf{x}_k\}) &\iff \mathbf{x}_k \in (V + \mathbf{x}_1) \setminus \{\mathbf{x}_k\} \\ &\iff \mathbf{x}_k - \mathbf{x}_1 \in V \setminus \{\mathbf{x}_k - \mathbf{x}_1\} \\ &\iff \mathbf{x}_k - \mathbf{x}_1 \in \text{lin}\ \{\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \dots, \mathbf{x}_{k-1} - \mathbf{x}_1, \\ &\qquad \dots, \mathbf{x}_{k+1} - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1\}\end{aligned}$$

Therefore, S is affinely dependent if and only if $\{\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1\}$ is linearly independent.

1.158 By the previous exercise, the set $S = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is affinely dependent if and only if the set $\{\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1\}$ is linearly dependent, so that there exist numbers $\alpha_2, \alpha_3, \dots, \alpha_n$, not all zero, such that

$$\alpha_2(\mathbf{x}_2 - \mathbf{x}_1) + \alpha_3(\mathbf{x}_3 - \mathbf{x}_1) + \cdots + \alpha_n(\mathbf{x}_n - \mathbf{x}_1) = \mathbf{0}$$

or

$$\alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \cdots + \alpha_n\mathbf{x}_n - \sum_{i=2}^{n} \alpha_i \mathbf{x}_1 = \mathbf{0}$$

Let $\alpha_1 = -\sum_{i=2}^n \alpha_i$. Then

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n = \mathbf{0}$$

and

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 0$$

as required.

1.159 Let

$$V = \operatorname{aff} S - \mathbf{x}_1 = \operatorname{aff} \{\mathbf{0}, \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1\}$$

Then

$$\operatorname{aff} S = \mathbf{x}_1 + V$$

If S is affinely independent, every $\mathbf{x} \in \operatorname{aff} S$ has a unique representation as

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{v}, \qquad \mathbf{v} \in V$$

with

$$\mathbf{v} = \alpha_2(\mathbf{x}_2 - \mathbf{x}_1) + \alpha_3(\mathbf{x}_3 - \mathbf{x}_1) + \cdots + \alpha_n(\mathbf{x}_n - \mathbf{x}_1)$$

so that

$$\mathbf{x} = \mathbf{x}_1 + \alpha_2(\mathbf{x}_2 - \mathbf{x}_1) + \alpha_3(\mathbf{x}_3 - \mathbf{x}_1) + \cdots + \alpha_n(\mathbf{x}_n - \mathbf{x}_1)$$

Define $\alpha_1 = 1 - \sum_{i=2}^n \alpha_i$. Then

$$\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n$$

with

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

$\mathbf{x}$ is a unique affine combination of the elements of S.

1.160 Assume that $x, y \in (a, b) \subseteq \Re$. This means that $a < x < b$ and $a < y < b$. For every $0 \le \alpha \le 1$

$$\alpha x + (1-\alpha)y > \alpha a + (1-\alpha)a$$

and

$$\alpha x + (1-\alpha)y < \alpha b + (1-\alpha)b$$

Therefore $a < \alpha x + (1-\alpha)y < b$ and $\alpha x + (1-\alpha)y \in (a,b)$. (a,b) is convex. Substituting $\leq$ for $<$ demonstrates that $[a,b]$ is convex.

Let S be an arbitrary convex set in $\Re$. Assume that S is not an interval. This implies that there exist numbers x, y, z such that $x < y < z$ and $x, z \in S$ while $y \notin S$. Define

$$\alpha = \frac{z-y}{z-x}$$

so that

$$1 - \alpha = \frac{y-x}{z-x}$$

Note that $0 \leq \alpha \leq 1$ and that

$$\alpha x + (1-\alpha)z = \frac{z-y}{z-x}x + \frac{y-x}{z-x}z = y \notin S$$

which contradicts the assumption that S is convex. We conclude that every convex set in $\Re$ is an interval. Note that S may be a hybrid interval such $(a,b]$ or $[a,b)$ as well as an open (a,b) or closed $[a,b]$ interval.

1.161 Let (N,w) be a TP-coalitional game. If $\text{core}(N,w) = \emptyset$ then it is trivially convex. Otherwise, assume $\text{core}(N,w)$ is nonempty and let $\mathbf{x}^1$ and $\mathbf{x}^2$ belong to $\text{core}(N,w)$. That is

$$\sum_{i \in S} x_i^1 \geq w(S) \qquad \text{for every } \ S \subseteq N$$
$$\sum_{i \in N} x_i^1 = w(N)$$

and therefore for any $0 \leq \alpha \leq 1$

$$\sum_{i \in S} \alpha x_i^1 \geq \alpha w(S) \qquad \text{for every } \ S \subseteq N$$
$$\sum_{i \in N} \alpha x_i^1 = \alpha w(N)$$

Similarly

$$\sum_{i \in S} (1-\alpha) x_i^2 \geq (1-\alpha) w(S) \qquad \text{for every } \ S \subseteq N$$
$$\sum_{i \in N} (1-\alpha) x_i^2 = (1-\alpha) w(N)$$

Summing these two systems

$$\sum_{i \in S} \alpha x_i^1 + (1-\alpha) x_i^2 \geq \alpha w(S) + (1-\alpha) w(S) = w(S) \qquad \text{for every } \ S \subseteq N$$
$$\sum_{i \in N} \alpha x_i^1 + (1-\alpha) x_i^2 = \alpha w(N) + (1-\alpha) w(N) = w(N)$$

That is, $\alpha x_i^1 + (1-\alpha) x_i^2$ belongs to $\text{core}(N,w)$.

1.162 Let $\mathfrak{C}$ be a collection of convex sets and let $\mathbf{x}$, $\mathbf{y}$ belong to $\bigcap_{S\in\mathfrak{C}} S$. for every $S \in \mathfrak{C}$, $\mathbf{x}, \mathbf{y} \in S$ and therefore $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S$ for all $0 \leq \alpha \leq 1$ (since S is convex). Therefore $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \bigcap_{S\in\mathfrak{C}} S$.

1.163 Fix some output y. Assume that $\mathbf{x}_1, \mathbf{x}_2 \in V(y)$. This implies that both $(y, -\mathbf{x}_1)$ and $(y, -\mathbf{x}_2)$ belong to the production possibility set Y. If Y is convex

$$\begin{aligned}
\alpha(y, -\mathbf{x}_1) + (1-\alpha)(y, -\mathbf{x}_2) &= (\alpha y + (1-\alpha)y, \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \\
&= (y, \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \in Y
\end{aligned}$$

for every $\alpha \in [0,1]$. This implies that $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \in V(y)$. Since the choice of y was arbitrary, this implies that $V(y)$ is convex for every y.

1.164 Assume S_1 and S_2 are convex sets. Let $S = S_1 + S_2$. Suppose $\mathbf{x}, \mathbf{y}$ belong to S. Then there exist $\mathbf{s}_1, \mathbf{t}_1 \in S_1$ and $\mathbf{s}_2, \mathbf{t}_2 \in S_2$ such that $\mathbf{x} = \mathbf{s}_1 + \mathbf{s}_2$ and $\mathbf{y} = \mathbf{t}_1 + \mathbf{t}_2$. For any $\alpha \in [0,1]$

$$\begin{aligned}
\alpha\mathbf{x} + (1-\alpha)\mathbf{y} &= \alpha\mathbf{s}_1 + \mathbf{s}_2 + (1-\alpha)\mathbf{t}_1 + \mathbf{t}_2 \\
&= \alpha\mathbf{s}_1 + (1-\alpha)\mathbf{t}_1 + \alpha\mathbf{s}_2 + (1-\alpha)\mathbf{t}_2 \in S
\end{aligned}$$

since $\alpha\mathbf{s}_1 + (1-\alpha)\mathbf{t}_1 \in S_1$ and $\alpha\mathbf{s}_2 + (1-\alpha)\mathbf{t}_2 \in S_2$. The argument readily extends to any finite number of sets.

1.165 Without loss of generality, assume that $n = 2$. Let $S = S_1 \times S_2 \subseteq X = X_1 \times X_2$. Suppose $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ belong to S. Then

$$\begin{aligned}
\alpha\mathbf{x} + (1-\alpha)\mathbf{y} &= \alpha(x_1, x_2) + (1-\alpha)(y_1, y_2) \\
&= (\alpha x_1, \alpha x_2) + ((1-\alpha)y_1, (1-\alpha)y_2) \\
&= (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2) \in S
\end{aligned}$$

1.166 Let $\alpha\mathbf{x}, \alpha\mathbf{y}$ be points in αS so that $\mathbf{x}, \mathbf{y} \in S$. Since S is convex, $\beta\mathbf{x} + (1-\beta)\mathbf{y} \in S$ for every $0 \leq \beta \leq 1$. Multiplying by α

$$\alpha(\beta\mathbf{x} + (1-\beta)\mathbf{y}) = \beta(\alpha\mathbf{x}) + (1-\beta)(\alpha\mathbf{y}) \in \alpha S$$

Therefore, αS is convex.

1.167 Combine Exercises 1.164 and 1.166.

1.168 The inclusion $S \subseteq \alpha S + (1-\alpha)S$ is true for any set (whether convex or not), since for every $\mathbf{x} \in S$

$$\mathbf{x} = \alpha\mathbf{x} + (1-\alpha)\mathbf{x} \in \alpha S + (1-\alpha)S$$

The reverse inclusion $\alpha S + (1-\alpha)S \subseteq S$ follows directly from the definition of convexity.

1.169 Given any two convex sets S and T in a linear space, the largest convex set contained in both is $S \cap T$; the smallest convex set containing both is $\operatorname{conv} S \cup T$. Therefore, the set of all convex sets is a lattice with

$$\begin{aligned}
S \wedge T &= S \cap T \\
S \vee T &= \operatorname{conv} S \cup T
\end{aligned}$$

The lattice is complete since every collection $\{S_i\}$ has a least upper bound $\operatorname{conv} \cup S_i$ and a greatest lower bound $\cap S_i$.

1.170 If a set contains all convex combinations of its elements, it contains all convex combinations of any two points, and hence is convex.

Conversely, assume that S is convex. Let $\mathbf{x}$ be a convex combination of elements in S, that is let

$$\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \ldots + \alpha_n\mathbf{x}_n$$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in S$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Re_+$ with $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$. We need to show that $\mathbf{x} \in S$.

We proceed by induction of the number of points n. Clearly, $\mathbf{x} \in S$ if $n = 1$ or $n = 2$. To show that it is true for $n = 3$, let

$$\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3$$

where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in S$ and $\alpha_1, \alpha_2, \alpha_3 \in \Re_+$ with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Assume that $\alpha_i > 0$ for all i (otherwise $n = 1$ or $n = 2$) so that $\alpha_1 < 1$. Rewriting

$$\begin{aligned}
\mathbf{x} &= \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 \\
&= \alpha_1\mathbf{x}_1 + (1-\alpha_1)\left(\frac{\alpha_2}{1-\alpha_1}\mathbf{x}_2 + \frac{\alpha_2}{1-\alpha_1}\mathbf{x}_3\right) \\
&= \alpha_1\mathbf{x}_1 + (1-\alpha_1)\mathbf{y}
\end{aligned}$$

where

$$\mathbf{y} = \left(\frac{\alpha_2}{1-\alpha_1}\mathbf{x}_2 + \frac{\alpha_2}{1-\alpha_1}\mathbf{x}_3\right)$$

$\mathbf{y}$ is a convex combination of two elements $\mathbf{x}_2$ and $\mathbf{x}_3$ since

$$\frac{\alpha_2}{1-\alpha_1} + \frac{\alpha_2}{1-\alpha_1} = \frac{\alpha_2 + \alpha_3}{1-\alpha_1} = 1$$

and $\alpha_2 + \alpha_3 = 1 - \alpha_1$. Hence $\mathbf{y} \in S$. Therefore $\mathbf{x}$ is a convex combination of two elements $\mathbf{x}_1$ and y and is also in S. Proceeding in this fashion, we can show that every convex combination belongs to S, that is conv $S \subseteq S$.

1.171 This is precisely analogous to Exercise 1.128. We observe that

1. conv S is a convex set.
2. if C is any convex set containing S, then conv $S \subseteq C$.

Therefore, conv S is the smallest convex set containing S.

1.172 Note first that $S \subseteq \text{conv } S$ for any set S. The converse for convex sets follows from Exercise 1.170.

1.173 Assume $\mathbf{x} \in \text{conv }(S_1 + S_2)$. Then, there exist numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ and vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in $S_1 + S_2$ such that

$$\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n$$

For every $\mathbf{x}_i$, there exists $\mathbf{x}_i^1 \in S_1$ and $\mathbf{x}_i^2 \in S_2$ such that

$$\mathbf{x}_i = \mathbf{x}_i^1 + \mathbf{x}_i^2$$

and therefore

$$\begin{aligned}\mathbf{x} &= \sum_{i=1}^n \alpha_i \mathbf{x}_i^1 + \sum_{i=1}^n \alpha_i \mathbf{x}_i^2 \\ &= \mathbf{x}^1 + \mathbf{x}^2\end{aligned}$$

where $\mathbf{x}^1 = \sum_{i=1}^n \alpha_i \mathbf{x}_i^1 \in S_1$ and $\mathbf{x}^2 = \sum_{i=1}^n \alpha_i \mathbf{x}_i^2 \in S_2$. Therefore $\mathbf{x} \in \text{conv } S_1 + \text{conv } S_2$.

Conversely, assume that $\mathbf{x} \in \text{conv } S_1 + \text{conv } S_2$. Then $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2$, where

$$\begin{aligned}\mathbf{x}^1 &= \sum_{i=1}^n \alpha_i x_i^1, \qquad \mathbf{x}_i^1 \in S_1 \\ \mathbf{x}^2 &= \sum_{j=1}^m \beta_j x_j^2, \qquad \mathbf{x}_i^2 \in S_2\end{aligned}$$

and

$$\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2 = \sum_{i=1}^n \alpha_i x_i^1 + \sum_{j=1}^m \beta_j x_j^2 \in \text{conv } (S_1 + S_2)$$

since $\mathbf{x}_i^1, \mathbf{x}_j^2 \in S_1 + S_2$ for every i and j.

1.174 The dimension of the input requirement set $V(y)$ is n. Its affine hull is $\Re^n$.

1.175 1. Let

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \ldots + \alpha_n \mathbf{x}_n \tag{1.7}$$

If $n > \dim S + 1$, the elements $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in S$ are affinely dependent (Exercise 1.157 and therefore there exist numbers $\beta_1, \beta_2, \ldots, \beta_n$, not all zero, such that (Exercise 1.158)

$$\beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \ldots + \beta_n \mathbf{x}_n = \mathbf{0} \tag{1.8}$$

and

$$\beta_1 + \beta_2 + \ldots + \beta_n = 0$$

2. Combining (1.7) and (1.8)

$$\begin{aligned}\mathbf{x} &= \mathbf{x} - t\mathbf{0} \\ &= \sum_{i=1}^n \alpha_i \mathbf{x}_i - t \sum_{i=1}^n \beta_i \mathbf{x}_i \\ &= \sum_{i=1}^n (\alpha_i - t\beta_i) \mathbf{x}_i \qquad (1.9)\end{aligned}$$

for any $t \in \Re$.

3. Let $t = \min_i \left\{ \frac{\alpha_i}{\beta_i} : \beta_i > 0 \right\} = \frac{\alpha_j}{\beta_j}$

We note that

- $t > 0$ since $\alpha_i > 0$ for every i.

- If $\beta_i > 0$, then $\alpha_i/\beta_i \geq \alpha_j/\beta_j \geq t$ and therefore $\alpha_i - t\beta_i \geq 0$
- If $\beta_i \leq 0$ then $\alpha_i - t\beta_i > 0$ for every $t > 0$.
- Therefore $\alpha_i - t\beta_i \geq 0$ for every t and
- $\alpha_i - t\beta_i = 0$ for $i = j$.

Therefore, (1.9) represents $\mathbf{x}$ as a convex combination of only $n-1$ points.

4. This process can be repeated until $\mathbf{x}$ is represented as a convex combination of at most $\dim S + 1$ elements.

1.176 Assume $\mathbf{x}$ is not an extreme point of S. Then there exists distinct $\mathbf{x}_1$ and $\mathbf{x}_2$ in S such that

$$\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$$

Without loss of generality, assume $\alpha \leq 1/2$ and let $\mathbf{y} = \mathbf{x}_2 - \mathbf{x}$. Then $\mathbf{x} + \mathbf{y} = \mathbf{x}_2 \in S$. Furthermore

$$\begin{aligned}
\mathbf{x} - \mathbf{y} &= \mathbf{x} - \mathbf{x}_2 + \mathbf{x} \\
&= 2\mathbf{x} - \mathbf{x}_2 \\
&= 2(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) - \mathbf{x}_2 \\
&= 2\alpha\mathbf{x}_1 + (1-2\alpha)\mathbf{x}_2 \in S
\end{aligned}$$

since $\alpha \leq 1/2$.

1.177 1. For any $\mathbf{x} = (x_1, x_2) \in C_2$, there exists some $\alpha_1 \in [0,1]$ such that

$$x_1 = \alpha_1 c + (1-\alpha_1)(-c) = (2\alpha_1 - 1)c$$

In fact, α_1 is defined by

$$\alpha_1 = \frac{x_1 + c}{2c}$$

Therefore (see Figure 1.5)

$$\begin{pmatrix} x_1 \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} c \\ c \end{pmatrix} + (1-\alpha_1)\begin{pmatrix} -c \\ c \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ -c \end{pmatrix} = \alpha_1 \begin{pmatrix} c \\ -c \end{pmatrix} + (1-\alpha_1)\begin{pmatrix} -c \\ -c \end{pmatrix}$$

Similarly $x_2 = \alpha_2 c + (1-\alpha_2)(-c)$ where

$$\alpha_2 = \frac{x_2 + c}{2c}$$

Therefore, for any $\mathbf{x} \in C_2$,

$$\begin{aligned}
\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \alpha_2 \begin{pmatrix} x_1 \\ c \end{pmatrix} + (1-\alpha_2)\begin{pmatrix} x_1 \\ -c \end{pmatrix} \\
&= \alpha_1\alpha_2 \begin{pmatrix} c \\ c \end{pmatrix} + (1-\alpha_1)\alpha_2 \begin{pmatrix} -c \\ c \end{pmatrix} \\
&\quad + \alpha_1(1-\alpha_2)\begin{pmatrix} c \\ -c \end{pmatrix} + (1-\alpha_1)(1-\alpha_2)\begin{pmatrix} -c \\ -c \end{pmatrix} \\
&= \beta_1 \begin{pmatrix} c \\ c \end{pmatrix} + \beta_2 \begin{pmatrix} -c \\ c \end{pmatrix} + \beta_3 \begin{pmatrix} c \\ -c \end{pmatrix} + \beta_4 \begin{pmatrix} -c \\ -c \end{pmatrix}
\end{aligned}$$

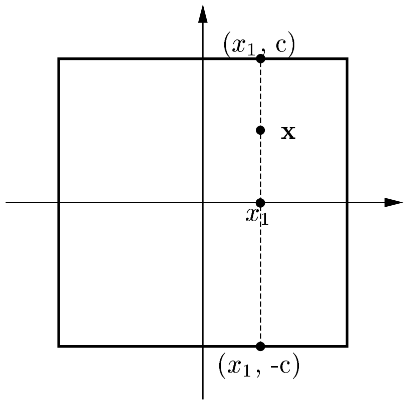

Figure 1.5: A cube in $\Re^2$

where $0 \leq \beta_i \leq 1$ and

$$\begin{aligned}\beta_1 + \beta_2 + \beta_3 + \beta_4 &= \alpha_1\alpha_2 + (1-\alpha_1)\alpha_2 + \alpha_1(1-\alpha_2) + (1-\alpha_1)(1-\alpha_2) \\ &= \alpha_1\alpha_2 + \alpha_2 - \alpha_1\alpha_2 + \alpha_1 - \alpha_1\alpha_2 + 1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 \\ &= 1\end{aligned}$$

That is

$$x \in \text{conv}\left\{ \begin{pmatrix} c \\ c \end{pmatrix}, \begin{pmatrix} -c \\ c \end{pmatrix}, \begin{pmatrix} c \\ -c \end{pmatrix}, \begin{pmatrix} -c \\ -c \end{pmatrix} \right\}$$

2. (a) For any point $(x_1, x_2, \ldots, x_{n-1}, c)$ which lies on face of the cube C_n, $(x_1, x_2, \ldots, x_{n-1}) \in C_{n-1}$ and therefore

$$(x_1, x_2, \ldots, x_{n-1}) \in \text{conv}\,\{\, \pm c, \pm c, \ldots, \pm c) \,\} \subseteq \Re^{n-1}$$

so that

$$\mathbf{x} \in \text{conv}\,\{\, (\pm c, \pm c, \ldots, \pm c, c) \,\} \subseteq \Re^n$$

Similarly, any point $(x_1, x_2, \ldots, x_{n-1}, -c)$ on the opposite face lies in the convex hull of the points $\{\, (\pm c, \pm c, \ldots, \pm c, -c) \,\}$.

(b) For any other point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in C_n$, let

$$\alpha_n = \frac{x_n + c}{2c}$$

so that

$$x_n = \alpha_n c + (1 - \alpha_n)(-c)$$

Then

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_{n-1} \\ x_n \end{pmatrix} = \alpha_n \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_{n-1} \\ c \end{pmatrix} + (1 - \alpha_n) \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_{n-1} \\ -c \end{pmatrix}$$

Hence

$$\mathbf{x} \in \text{conv}\,\{\,(\pm c, \pm c, \dots, \pm c)\,\} \subset \Re^n$$

In other words

$$C_n \subseteq \text{conv}\,\{\,(\pm c, \pm c, \dots, \pm c)\,\} \subset \Re^n$$

3. Let E denote the set of points of the form $\{\,(\pm c, \pm c, \dots, \pm c)\,\} \subseteq \Re^n$. Clearly, every point in E is an extreme point of C_n. Conversely, we have shown that $C_n \subseteq \text{conv}\, E$. Therefore, no point $\mathbf{x} \in C^n \setminus E$ can be an extreme point of C^n. E is the set of extreme points of C^n.
4. Since C^n is convex, and $E \subset C_n$, conv $E \subseteq C^n$. Consequently, $C^n = \text{conv}\, E$.

1.178 Let $\mathbf{x}$, $\mathbf{y}$ belong to $S \setminus F$ is convex. For any $\alpha \in [0, 1]$

- $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S$ since S convex
- $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \notin F$ since F is a face

Thus $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S \setminus F$ which is convex.

1.179 1. Trivial.

2. Let $\{F_i\}$ be a collection of faces of S and let $F = \bigcup F_i$. Choose any $\mathbf{x}, \mathbf{y} \in S$. If the line segment between $\mathbf{x}$ and $\mathbf{y}$ intersects F, then it intersects some face F_i which implies that $\mathbf{x}, \mathbf{y} \in F_i$. Therefore, $\mathbf{x}, \mathbf{y} \in F = \bigcup F_i$.
3. Let $\{F_i\}$ be a collection of faces of S and let $F = \bigcap F_i$. Choose any $\mathbf{x}, \mathbf{y} \in S$. if the line segment between $\mathbf{x}$ and $\mathbf{y}$ intersects F, then it intersects every face F_i which implies that $\mathbf{x}, \mathbf{y} \in F_i$ for every i. Therefore, $\mathbf{x}, \mathbf{y} \in F = \bigcup F_i$.
4. Let $\mathfrak{F}$ be the collection of all faces of S. This is partially ordered by inclusion. By the previous result, every nonempty subcollection $\mathfrak{G}$ has a least upper bound $(\bigcap_{F \in \mathfrak{G}} F)$. Hence $\mathfrak{F}$ is a complete lattice (Exercise 1.47).

1.180 Let S be a polytope. Then $S = \text{conv}\,\{\,\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\,\}$. Note that every extreme point belongs to $\{\,\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\,\}$. Now choose the smallest subset whose convex hull is still S, that is delete elements which can be written as convex combinations of other elements. Suppose the minimal subset is $\{\,\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\,\}$. We claim that each of these elements is an extreme point of S, that is $\{\,\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\,\} = E$.

Assume not, that is assume that $\mathbf{x}_m$ is not an extreme point so that there exists $\mathbf{x}, \mathbf{y} \in S$ with

$$\mathbf{x}_m = \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \qquad \text{with } 0 < \alpha < 1 \tag{1.10}$$

Since $\mathbf{x}, \mathbf{y} \in \text{conv}\,\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$

$$\mathbf{x} = \sum_{i=1}^{m} \alpha_i \mathbf{x}_i \qquad \mathbf{y} = \sum_{i=1}^{m} \beta \mathbf{x}_i$$

Substituting in (1.10), we can write $\mathbf{x}_m$ as a convex combination of $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$.

$$\mathbf{x}_m = \sum_{i=1}^{m} \big(\alpha\alpha_i + (1-\alpha)\beta_i\big)\mathbf{x}_i = \sum_{i=1}^{m} \gamma_i \mathbf{x}_i$$

where

$$\gamma_i = \alpha\alpha_i + (1-\alpha)\beta_i$$

Note that $0 \leq \gamma_i \leq 1$, so that either $\gamma_m < 1$ or $\gamma_m = 1$. We show that both cases lead to a contradiction.

- $\gamma_m < 1$. Then

$$\mathbf{x}_m = \frac{1}{1-\gamma_m} \sum_{i=1}^{m-1} \big(\alpha\alpha_i + (1-\alpha)\beta_i\big)\mathbf{x}_i$$

  which contradicts the minimality of the set $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$.

- $\gamma_m = 1$. Then $\gamma_i = 0$ for every $i \neq m$. That is

$$\alpha\alpha_i + (1-\alpha)\beta_i = 0 \qquad \text{for every } \ i \neq m$$

  which implies that $\alpha_i = \beta_i \qquad$ for every $i \neq m$ and therefore $\mathbf{x} = \mathbf{y}$.

Therefore, if $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ is a minimal spanning set, every point must be an extreme point.

1.181 Assume to the contrary that one of the vertices is not an extreme point of the simplex. Without loss of generality, assume this is $\mathbf{x}_1$. Then, there exist distinct $\mathbf{y}, \mathbf{z} \in S$ and $0 < \alpha < 1$ such that

$$\mathbf{x}_1 = \alpha\mathbf{y} + (1-\alpha)\mathbf{z} \tag{1.11}$$

Now, since $\mathbf{y} \in S$, there exist $\beta_1, \beta_2, \dots, \beta_n$ such that

$$\mathbf{y} = \sum_{i=1}^n \beta_i\mathbf{x}_i, \qquad \sum_{i=1}^n \beta_i = 1$$

Similarly, there exist $\delta_1, \delta_2, \dots, \delta_n$ such that

$$\mathbf{z} = \sum_{i=1}^n \delta_i\mathbf{x}_i, \qquad \sum_{i=1}^n \delta_i = 1$$

Substituting in (1.11)

$$\begin{aligned}\mathbf{x}_1 &= \alpha\sum_{i=1}^n \beta_i\mathbf{x}_i + (1-\alpha)\sum_{i=1}^n \delta_i\mathbf{x}_i \\ &= \sum_{i=1}^n \big(\alpha\beta_i + (1-\alpha)\delta_i\big)\mathbf{x}_i\end{aligned}$$

Since $\sum_{i=1}^n \big(\alpha\beta_i + (1-\alpha)\delta_i\big) = \alpha\sum_{i=1}^n \beta_i + (1-\alpha)\sum_{i=1}\delta_i = 1$

$$\mathbf{x}_1 = \sum_{i=1}^n \big(\alpha\beta_i + (1-\alpha)\delta_i\big)\mathbf{x}_i$$

Subtracting, this implies

$$\mathbf{0} = \sum_{i=2}^n \big(\alpha\beta_i + (1-\alpha)\delta_i\big)(\mathbf{x}_i - \mathbf{x}_1)$$

This establishes that the set $\{\mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_1, \dots, \mathbf{x}_n - \mathbf{x}_1\}$ is linearly dependent and therefore $E = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is affinely dependent (Exercise 1.157). This contradicts the assumption that S is a simplex.

1.182 Let n be the dimension of a convex set S in a linear space X. Then $n = \dim \operatorname{aff} S$ and there exists a set $\{ \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1} \}$ of affinely independent points in S. Define

$$S' = \operatorname{conv} \{ \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n+1} \}$$

Then S' is an n-dimensional simplex contained in S.

1.183 Let $\mathbf{w} = (w(\{1\}), w(\{2\}), \dots, w(\{n\}))$ denote the vector of individual worths and let s denote the surplus to be distributed, that is

$$s = w(N) - \sum_{i \in N} w(\{i\})$$

$s > 0$ if the game is essential. For each player $i = 1, 2, \dots, n$, let

$$\mathbf{y}^i = \mathbf{w} + s\mathbf{e}_i$$

be the outcome in which player i receives the entire surplus. ($\mathbf{e}_i$ is the ith unit vector.) Note that

$$y^i_j = \begin{cases} w(\{i\}) + s & j = i \\ w(\{i\}) & j \neq i \end{cases}$$

Each $\mathbf{y}^i$ is an imputation since $y^i_j \geq w(\{j\})$ and

$$\sum_{j \in N} y^i_j = \sum_{j \in N} w(\{j\}) + s = w(N)$$

Therefore $\{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^n \} \subseteq I$. Since I is convex (why ?), $S = \operatorname{conv} \{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^n \} \subseteq I$. Further, for every $i, j \in N$ the vectors

$$\mathbf{y}^i - \mathbf{y}^j = s(\mathbf{e}_i - \mathbf{e}_j)$$

are linearly independent. Therefore S is an $n-1$-dimensional simplex in $\Re^n$.

For any $\mathbf{x} \in I$ define

$$\alpha_i = \frac{x_i - w(\{i\})}{s}$$

so that

$$x_j = w(\{j\}) + \alpha_j s$$

Since $\mathbf{x}$ is an imputation

- $\alpha_i \geq 0$
- $\sum_{i \in N} \alpha_i = \left(\sum_{i \in N} x_i - \sum_{i \in N} w(\{i\}) \right) / s = 1$

We claim that $\mathbf{x} = \sum_{i \in N} \alpha_i \mathbf{y}^i$ since for each $j = 1, 2, \dots, n$

$$\begin{aligned} \sum_{i \in N} \alpha_i y^i_j &= \sum_{i \in N} \alpha_i w(\{j\}) + \alpha_j s \\ &= w(\{j\}) + \alpha_j s \\ &= x_j \end{aligned}$$

Therefore $\mathbf{x} \in \operatorname{conv} \{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^n \} = S$, that is $I \subseteq S$. Since we previously showed that $S \subseteq I$, we have established that $I = S$, which is an $n-1$ dimensional simplex in $\Re^n$.

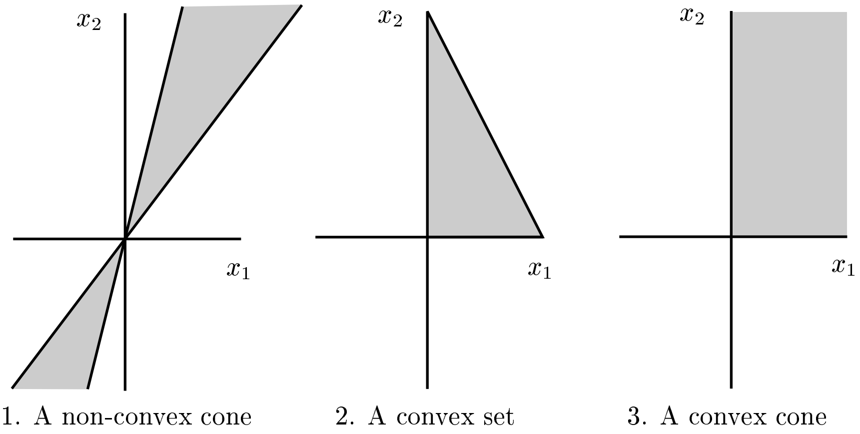

Figure 1.6: A cone which is not convex, a convex set and a convex cone

1.184 See Figure 1.6.

1.185 Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ belong to $\Re^n_+$, which means that $x_i \geq 0$ for every i. For every $\alpha > 0$

$$\alpha\mathbf{x} = (\alpha\mathbf{x}_1, \alpha\mathbf{x}_2, \ldots, \alpha\mathbf{x}_n)$$

and $\alpha x_i \geq 0$ for every i. Therefore $\alpha\mathbf{x} \in \Re^n_+$. $\Re^n_+$ is a cone in $\Re^n$.

1.186 Assume

$$\alpha\mathbf{x} + \beta\mathbf{y} \in S \text{ for every } \mathbf{x}, \mathbf{y} \in S \text{ and } \alpha, \beta \in \Re_+ \tag{1.12}$$

Letting $\beta = 0$, this implies that

$$\alpha\mathbf{x} \in S \text{ for every } \mathbf{x} \in S \text{ and } \alpha \in \Re_+$$

so that S is a cone. To show that S is convex, let $\mathbf{x}$ and $\mathbf{y}$ be any two elements in S. For any $\alpha \in [0, 1]$, (1.12) implies that

$$\alpha\mathbf{x} + (1 - \alpha)\mathbf{y} \in S$$

Therefore S is convex.

Conversely, assume that S is a convex cone. For any $\alpha, \beta \in \Re_+$ and $\mathbf{x}, \mathbf{y} \in S$

$$\frac{\alpha}{\alpha + \beta}\mathbf{x} + \frac{\beta}{\alpha + \beta}\mathbf{y} \in S$$

and therefore

$$\alpha\mathbf{x} + \beta\mathbf{y} \in S$$

1.187 Assume S satisfies

1. $\alpha S \subseteq S$ for every $\alpha \geq 0$
2. $S + S \subseteq S$

By (1), S is a cone. To show that it is convex, let $\mathbf{x}$ and $\mathbf{y}$ belong to S. By (1), $\alpha\mathbf{x}$ and $(1-\alpha)\mathbf{y}$ belong to S, and therefore $\alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ belongs to S by (2). S is convex.

Conversely, assume that S is a convex cone. Then

$$\alpha S \subseteq S \qquad \text{for every} \quad \alpha \geq 0$$

Let $\mathbf{x}$ and $\mathbf{y}$ be any two elements in S. Since S is convex, $z = \alpha\frac{1}{2}\mathbf{x} + (1-\alpha)\frac{1}{2}\mathbf{y} \in S$ and since it is a cone, $2z = \mathbf{x} + \mathbf{y} \in S$. Therefore

$$S + S \subseteq S$$

1.188 We have to show that Y is convex cone. By assumption, Y is convex. To show that Y is a cone, let $\mathbf{y}$ be any production plan in Y. By convexity

$$\alpha\mathbf{y} = \alpha\mathbf{y} + (1-\alpha)\mathbf{0} \in Y \text{ for every } \ 0 \leq \alpha \leq 1$$

Repeated use of additivity ensures that

$$\alpha\mathbf{y} \in Y \text{ for every } \ \alpha = 1, 2, \dots$$

Combining these two conclusions implies that

$$\alpha\mathbf{y} \in Y \text{ for every } \ \alpha \geq 0$$

1.189 Let $\mathcal{S} \subset \mathcal{G}^N$ denote the set of all superadditive games. Let $w^1, w^2 \in S$ be two superadditive games. Then, for all distinct coalitions $S, T \subset N$ with $S \cap T = \emptyset$

$$w^1(S \cup T) \geq w^1(S) + w^1(T)$$
$$w^2(S \cup T) \geq w^2(S) + w^2(T)$$

Adding

$$\begin{aligned}
(w^1 + w^2)(S \cup T) &= w^1(S \cup T) + w^2(S \cup T) \\
&\geq w^1(S) + w^2(S) + w^1(T) + w^2(T) \\
&= (w^1 + w^2)(S) + (w^1 + w^2)(T)
\end{aligned}$$

so that $w^1 + w^2$ is superadditive. Similarly, we can show that αw^1 is superadditive for all $\alpha \in \Re^+$. Hence $\mathcal{S}$ is a convex cone in $\mathcal{G}^N$.

1.190 Let $\mathbf{x}$ belong to $\bigcap_{i=1}^n S_i$. Then $\mathbf{x} \in S_i$ for every i. Since each S_i is a cone, $\alpha\mathbf{x} \in S_i$ for every $\alpha \geq 0$ and therefore $\alpha\mathbf{x} \in \bigcap_{i=1}^n S_i$.

Let $S = S_1 + S_2 + \cdots + S_n$ and assume $\mathbf{x}$ belongs to S. Then there exist $\mathbf{x}_i \in S_i$, $i = 1, 2, \dots, n$ such that

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n$$

For any $\alpha \geq 0$

$$\begin{aligned}
\alpha\mathbf{x} &= \alpha(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n) \\
&= \alpha\mathbf{x}_1 + \alpha\mathbf{x}_2 + \cdots + \alpha\mathbf{x}_n \in S
\end{aligned}$$

since $\alpha\mathbf{x}_i \in S_i$ for every i.

1.191 1. Suppose that $\mathbf{y} \in Y$. Then, there exist $\alpha_, \alpha_2, \dots, \alpha_8 \geq 0$ such that

$$\mathbf{y} = \sum_{i=1}^{8} \alpha_i \mathbf{y}_i$$

and for the first commodity

$$\mathbf{y}_1 = \sum_{i=1}^{8} \alpha_i y_{i1}$$

If $\mathbf{y} \neq 0$, at least one of the $\alpha_i > 0$ and hence $\mathbf{y}_1 < 0$ since $y_{i1} < 0$ for $i = 1, 2, \dots, 8$.

2. Free disposal requires that $\mathbf{y} \in Y, \mathbf{y}' \leq \mathbf{y} \implies \mathbf{y}' \in Y$. Consider the production plan $\mathbf{y}' = (-2, -2, -2, -2)$. Note that $\mathbf{y}' \leq \mathbf{y}_3$ and $\mathbf{y}' \leq \mathbf{y}_6$. Suppose that $\mathbf{y}' \in Y$. Then there exist $\alpha_1, \alpha_2, \dots, \alpha_8 \geq 0$ such that

$$\mathbf{y} = \sum_{i=1}^{8} \alpha_i \mathbf{y}_i$$

For the third commodity (component), we have

$$4\alpha_1 + 3\alpha_2 + 3\alpha_3 + 3\alpha_4 + 12\alpha_5 - 2\alpha_6 + 5\alpha_8 = -2 \tag{1.13}$$

and for the fourth commodity

$$2\alpha_2 - 1\alpha_3 + 1\alpha_4 + 5\alpha_6 + 10\alpha_7 - 2\alpha_8 = -2 \tag{1.14}$$

Adding to (1.14) to (1.13) gives

$$4\alpha_1 + 5\alpha_2 + 2\alpha_3 + 4\alpha_4 + 12\alpha_5 + 3\alpha_6 + 10\alpha_7 + 3\alpha_8 = -4$$

which is impossible given that $\alpha_i \geq 0$. Therefore, we conclude that $\mathbf{y}' \notin Y$.

3.

$$\begin{aligned}
\mathbf{y}_2 = (-7, -9, 3, 2) &\geq (-8, -13, 3, 1) = \mathbf{y}_4 \\
3\mathbf{y}_1 = (-9, -18, 12, 0) &\geq (-11, -19, 12, 0) = \mathbf{y}_5 \\
\mathbf{y}_7 = (-8, -5, 0, 10) &\geq (-8, -6, -4, 10) = 2\mathbf{y}_6 \\
2\mathbf{y}_3 = (-2, -4, 6, -2) &\geq (-2, -4, 5, -2) = \mathbf{y}_8
\end{aligned}$$

4.

$$\begin{aligned}
2\mathbf{y}_3 + \mathbf{y}_7 &= (-2, -4, 6, -2) + (-8, -5, 0, 10) \\
&= (-10, -9, 6, 8) \\
&\geq (-14, -18, 6, 4) = 2\mathbf{y}_2
\end{aligned}$$

$$\begin{aligned}
20\mathbf{y}_3 + 2\mathbf{y}_7 &= 20(-1, -2, 3, -1) + 2(-8, -5, 0, 10) \\
&= (-20, -40, 60, -20) + (-16, -10, 0, 20) \\
&= (-36, -50, 60, 0) \\
&\geq (-45, -90, 60, 0) = 15\mathbf{y}_1
\end{aligned}$$

5. $\text{Eff}(Y) = \text{cone}\,\{\,\mathbf{y}_3, \mathbf{y}_7\,\}$.

1.192 This is precisely analogous to Exercise 1.128. We observe that

1. cone S is a convex cone.
2. if C is any convex cone containing S, then conv $S \subseteq C$.

Therefore, cone S is the smallest convex cone containing S.

1.193 For any set S, $S \subseteq$ cone S. If S is a convex cone, Exercise 1.186 implies that cone $S \subseteq S$.

1.194 1. If $n > m = \dim \text{cone}\, S = \dim \text{lin}\, S$, the elements $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in S$ are linearly dependent and therefore there exist numbers $\beta_1, \beta_2, \dots, \beta_n$, not all zero, such that (Exercise 1.134)

$$\beta_1 \mathbf{x}_1 + \beta_2 \mathbf{x}_2 + \ldots + \beta_n \mathbf{x}_n = \mathbf{0} \tag{1.15}$$

2. Combining (1.14) and (1.15)

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} - t\mathbf{0} \\
&= \sum_{i=1}^{n} \alpha_i \mathbf{x}_i - t \sum_{i=1}^{n} \beta_i \mathbf{x}_i \\
&= \sum_{i=1}^{n} (\alpha_i - t\beta_i) \mathbf{x}_i
\end{aligned} \tag{1.16}$$

for any $t \in \Re$.

3. Let $t = \min_i \left\{ \frac{\alpha_i}{\beta_i} : \beta_i > 0 \right\} = \frac{\alpha_j}{\beta_j}$

We note that

- $t > 0$ since $\alpha_i > 0$ for every i.
- If $\beta_i > 0$, then $\alpha_i/\beta_i \geq \alpha_j/\beta_j \geq t$ and therefore $\alpha_i - t\beta_i \geq 0$.
- If $\beta_i \leq 0$ then $\alpha_i - t\beta_i > 0$ for every $t > 0$.
- Therefore $\alpha_i - t\beta_i \geq 0$ for every t and
- $\alpha_i - t\beta_i = 0$ for $i = j$.

Therefore, (1.16) represents $\mathbf{x}$ as a nonnegative combination of only $n-1$ points.

4. This process can be repeated until $\mathbf{x}$ is represented as a convex combination of at most m points.

1.195 1. The affine hull of $\tilde{S}$ is parallel to the affine hull of S. Therefore

$$\dim S = \dim \text{aff}\, S = \dim \text{aff}\, \tilde{S}$$

Since $\begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \notin \text{aff}\, \tilde{S}$,

$$\dim \text{cone}\, \tilde{S} = \dim \text{aff}\, \tilde{S} + 1 = \dim S + 1$$

2. For every $\mathbf{x} \in \operatorname{conv} S$, $\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \in \operatorname{conv} \tilde{S}$ and there exist (Exercise 1.194) $m+1$ points $\begin{pmatrix} \mathbf{x}_i \\ 1 \end{pmatrix} \in \tilde{S}$ such that

$$\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} \in \operatorname{conv} \left\{ \begin{pmatrix} \mathbf{x}_1 \\ 1 \end{pmatrix}, \begin{pmatrix} \mathbf{x}_2 \\ 1 \end{pmatrix}, \dots \begin{pmatrix} \mathbf{x}_{m+1} \\ 1 \end{pmatrix} \right\}$$

This implies that

$$\mathbf{x} \in \operatorname{conv} \{ \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m+1} \}$$

with $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m+1} \in S$.

1.196 A subsimplex with precisely one distinguished face is completely labeled. Suppose a subsimplex has more than one distinguished face. This means that it has vertices labeled $1, 2, \dots, n$. Since it has $n+1$ vertices, one of these labels must be repeated (twice). The distinguished faces lie opposite the repeated vertices. There are precisely two distinguished faces.

1.197
1. $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \geq 0$.
2. $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = 0$ if and only if $\mathbf{x} - \mathbf{y} = \mathbf{0}$, that is $\mathbf{x} = \mathbf{y}$.
3. Property (3) ensures that $\|-\mathbf{x}\| = \|\mathbf{x}\|$ and therefore $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$ so that

$$\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\| = \rho(\mathbf{y}, \mathbf{x})$$

4. For any $\mathbf{z} \in X$

$$\begin{aligned} \rho(\mathbf{x}, \mathbf{y}) &= \|\mathbf{x} - \mathbf{y}\| \\ &= \|\mathbf{x} - \mathbf{z} + \mathbf{z} - \mathbf{y}\| \\ &\leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\| \\ &= \rho(\mathbf{x}, \mathbf{z}) + \rho(\mathbf{z}, \mathbf{y}) \end{aligned}$$

Therefore $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ satisfies the properties required of a metric.

1.198 Clearly $\|\mathbf{x}\|_\infty \geq 0$ and $\|\mathbf{x}\|_\infty = 0$ if and only if $\mathbf{x} = 0$. Thirdly

$$\|\alpha \mathbf{x}\| = \max_{i=1}^{n} |\alpha x_i| = |\alpha| \max_{i=1}^{n} |x_i| = |\alpha| \, \|\mathbf{x}\|$$

To prove the triangle inequality, we note that for any $x_i, y_i \in \Re$

$$\max(x_i + y_i) \leq \max x_i + \max y_i$$

Therefore

$$\|\mathbf{x}\| = \max_{i=1}^{n} (x_i + y_i) \leq \max_{i=1}^{n} x_i + \max_{i=1}^{n} y_i = \|\mathbf{x}\| + \|\mathbf{y}\|$$

1.199 Suppose that producing one unit of good 1 requires two units of good 2 and three units of good 3. The production plan is $(1, -2, -3)$ and the average net output, -2, is negative. A norm is required to be nonnegative. Moreover, the quantities of inputs and outputs may balance out yielding a zero average. That is, $(\sum_{i=1}^{n} y_i)/n = 0$ does not imply that $y_i = 0$ for all i.

1.200

$$\begin{aligned}\|\mathbf{x}\| - \|\mathbf{y}\| &= \|\mathbf{x} - \mathbf{y} + \mathbf{y}\| - \|\mathbf{y}\| \\ &\leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\| - \|\mathbf{y}\| \\ &= \|\mathbf{x} - \mathbf{y}\|\end{aligned}$$

1.201 Using the previous exercise

$$\|\mathbf{x}_n\| - \|\mathbf{x}\| \leq \|\mathbf{x}_n - \mathbf{x}\| \to 0$$

1.202 First note that each term $\mathbf{x}_n + \mathbf{y}_n \in X$ by linearity. Similarly, $\mathbf{x} + \mathbf{y} \in X$. Fix some $\epsilon > 0$. There exists some $N_{\mathbf{x}}$ such that $\|\mathbf{x}_n - \mathbf{x}\| < \epsilon$ for all $n \geq N_{\mathbf{x}}$. Similarly, there exists some $N_{\mathbf{y}}$ such that $\|\mathbf{y}_n - \mathbf{y}\| < \epsilon$ for all $n \geq N_{\mathbf{y}}$. For all $n \geq \max\{ N_{\mathbf{x}}, N_{\mathbf{y}} \}$,

$$\begin{aligned}\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x} + \mathbf{y})\| &= \|(\mathbf{x}_n - \mathbf{x}) + (\mathbf{y}_n - \mathbf{y})\| \\ &\leq \|\mathbf{x}_n - \mathbf{x}\| + \|\mathbf{y}_n - \mathbf{y}\| \\ &< \epsilon\end{aligned}$$

Similarly, for every $n \geq N_{\mathbf{x}}$

$$\begin{aligned}\|\alpha\mathbf{x}_n - \alpha\mathbf{x}\| &= |\alpha| \, \|\mathbf{x}_n - \mathbf{x}\| \\ &\leq |\alpha| \, \epsilon/2 \\ &\to 0 \text{ as } \epsilon \to 0\end{aligned}$$

1.203 Let $\mathbf{x}_n$ be a sequence in $S + T$ converging to $\mathbf{x}$. For every n, there exists $\mathbf{y}_n \in S$ and $\mathbf{z}_n \in T$ such that $\mathbf{x}_n = \mathbf{y}_n + \mathbf{z}_n$. Since T is compact, there exists a subsequence $\mathbf{z}_m$ converging to $\mathbf{z} \in T$. Let $\mathbf{y} = \lim y_m$. Then $\mathbf{y} \in S$ since S is closed. By the previous exercise, $\mathbf{y}_m + \mathbf{z}_m \to \mathbf{y} + \mathbf{z}$. By assumption, $\mathbf{y}_m + \mathbf{z}_m \to \mathbf{x}$ so that $\mathbf{x} = \mathbf{y} + \mathbf{z} \in S + T$. $S + T$ is closed.

1.204 Yes. Apply Exercise 1.202.

1.205 The nth partial sum of the series is

$$\mathbf{s}_n = \mathbf{x} + \beta\mathbf{x} + \beta^2\mathbf{x} + \cdots + \beta^{n-1}\mathbf{x}$$

Multiplying this equation by β gives

$$\beta\mathbf{s}_n = \beta\mathbf{x} + \beta^2\mathbf{x} + \beta^3\mathbf{x} + \cdots + \beta^n\mathbf{x}$$

Subtracting this equation from the previous one and canceling common terms gives

$$(1 - \beta)\mathbf{s}_n = \mathbf{x} - \beta^n\mathbf{x} = (1 - \beta^n)\mathbf{x}$$

Provided that $\beta \neq 1$

$$\mathbf{s}_n = \frac{\mathbf{x} - \beta^n\mathbf{x}}{1 - \beta} = \frac{\mathbf{x}}{1 - \beta} - \frac{\beta^n\mathbf{x}}{1 - \beta} \tag{1.17}$$

If $\beta < 1$, then $\beta^n \to 0$ (Exercise 1.102) and therefore s^n converges to $\mathbf{x}/(1 - \beta)$.

1.206

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

is a geometric series $1 + \beta + \beta^2 + \beta^3 + \dots$ with $\beta = 1/2$. The series converges (Exercise 1.205) to

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1 - \beta} = \frac{1}{1 - \frac{1}{2}} = 2$$

1.207 The present value of the n payments is the nth partial sum of the geometric series $x + \beta x + \beta^2 x + \beta^3 x + \dots$ which (using (1.17)) is given by

$$\text{Present value} = s_n = \frac{x - \beta^n x}{1 - \beta}$$

1.208 By Exercise 1.93, there exists an open set $T \supseteq S_1$ such that $T \cap S_2 = \emptyset$. For every $\mathbf{x} \in S_1$, there exists an open ball $B(\mathbf{x})$ such that $B(\mathbf{x}) \subseteq T$ and therefore $B(\mathbf{x}) \cap S_2 = \emptyset$. The collection $\{ B(\mathbf{x}) \}$ of open balls is an open cover for S_1. Since S_1 is compact there exists a finite subcover, that is there exists points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ in S_1 such that

$$S_1 \subseteq \bigcup_{i=1}^{n} B(\mathbf{x}_i)$$

Furthermore, for every $\mathbf{x}_i$, there exists r_n such that

$$B(\mathbf{x}_i) = \mathbf{x}_i + r_n B$$

where B is the unit ball. Let $r = \min r_n$. $U = rB$ is the required neighborhood.

1.209 Clearly $X \times Y$ is a normed linear space. To show that it is complete, let $(\mathbf{z}^n)$ be a Cauchy sequence in $X \times Y$ where $\mathbf{z}^n = (\mathbf{x}^n, \mathbf{y}^n)$. For every $\epsilon > 0$, there exists some N such that

$$\|\mathbf{z}^n - \mathbf{z}^m\| = \max\{ \|\mathbf{x}^n - \mathbf{x}^m\|, \|\mathbf{y}^n - \mathbf{y}^m\| \} < \epsilon$$

for every $n, m \geq N$. This implies that $(\mathbf{x}^n)$ and $(\mathbf{y}^n)$ are Cauchy sequences in X and Y respectively. Since X and Y are complete, both sequences converge. That is, there exists $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ such that $\|\mathbf{x}^n - \mathbf{x}\| \to 0$ and $\|\mathbf{y}^n - \mathbf{y}\| \to 0$. In other words, given $\epsilon > 0$ there exists N such that $\|\mathbf{x}^n - \mathbf{x}\| < \epsilon$ and $\|\mathbf{y}^n - \mathbf{y}\| < \epsilon$ for every $n \geq N$. Let $\mathbf{z} = (\mathbf{x}, \mathbf{y})$. Then, for every $n \geq N$

$$\|\mathbf{z}^n - \mathbf{z}\| = \max\{ \|\mathbf{x}^n - \mathbf{x}\|, \|\mathbf{y}^n - \mathbf{y}\| \} < \epsilon$$

$\mathbf{z}^n \to \mathbf{z}$.

1.210

1. By assumption, for every $m = 1, 2, \dots$, there exists a point $\mathbf{y}^m$ such that

$$\|\mathbf{y}\| < \frac{1}{m} \left(\sum_{i=1}^{n} |\alpha_i| \right)$$

where

$$\mathbf{y} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \dots + \alpha_n \mathbf{x}_n$$

Let $s^m = \sum_{i=1}^{n} |\alpha_i|$. By assumption $s^m > m \|\mathbf{y}^m\| \geq 0$. Define

$$\mathbf{x}^m = \frac{1}{s^m} \mathbf{y}^m$$

Then

$$\mathbf{x}^m = \beta_1^m \mathbf{x}_1 + \beta_2^m \mathbf{x}_2 + \dots + \beta_n^m \mathbf{x}_n$$

where $\beta_i^m = \alpha_i^m / s^m$, $\sum_{i=1}^{n} |\beta_i^m| = 1$ and $\|\mathbf{x}^m\| < \frac{1}{m}$ for every $n = 1, 2, \dots$. Consequently $\|\mathbf{x}^m\| \to 0$.

2. Since $\sum_{i=1}^n |\beta_i^m| = 1$, $|\beta_i^m| \leq 1$ for every i. Consequently, for every coordinate i, the sequence (β_i^m) is bounded. By the Bolzano-Weierstrass theorem (Exercise 1.119), the sequence (β_1^m) has a convergent subsequence with $\beta_1^m \to \beta_1$. Let $\mathbf{x}^{m,1}$ denote the corresponding subsequence of $\mathbf{x}^m$.

   Similarly, $\beta_2^{m,1}$ has a subsequence converging to β_2. Let $(\mathbf{x}^{m,2})$ denote the corresponding subsequence of $(\mathbf{x}^m)$. Proceeding coordinate by coordinate, we obtain a subsequence $(\mathbf{x}^{m,n})$ where each term is

$$\mathbf{x}^{m,n} = \beta^{m,n}\mathbf{x}_1 + \beta^{m,n}\mathbf{x}_2 + \cdots + \beta^{m,n}\mathbf{x}_n$$

   and each coefficient converges $\beta_i^{m,n} \to \beta_i$. Let

$$\mathbf{x} = \beta_1\mathbf{x}_1 + \beta_2\mathbf{x}_2 + \cdots + \beta_2\mathbf{x}_n$$

   Then $\mathbf{x}^{m,n} \to \mathbf{x}$ (Exercise 1.202).

3. Since $\sum_{i=1}^n |\beta_i^m| = 1$ for every m, $\sum_{i=1}^n |\beta_i| = 1$. Consequently, at least one of the coefficients $\beta_i \neq 0$. Since $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ are linearly independent, $\mathbf{x} \neq \mathbf{0}$ (Exercise 1.133) and therefore $\|\mathbf{x}\| \neq 0$. But $(\mathbf{x}^{m,n})$ is a subsequence of $(\mathbf{x}^m)$. This contradicts the earlier conclusion (part 1) that $\|\mathbf{x}^m\| \to 0$.

1.211 1. Let X be a normed linear space X of dimension n and let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ be a basis for X. Let $(\mathbf{x}^m)$ be a Cauchy sequence in X. Each term $\mathbf{x}^m$ has a unique representation

$$\mathbf{x}^m = \alpha_1^m\mathbf{x}_1 + \alpha_2^m\mathbf{x}_2 + \cdots + \alpha_n^m\mathbf{x}_n$$

   We show that each of the sequences α_i^m is a Cauchy sequence in $\Re$.

   Since $\mathbf{x}^m$ is a Cauchy sequence, for every $\epsilon > 0$ there exists an N such that $\|\mathbf{x}^m - \mathbf{x}^r\| < \epsilon$ for all $m, r \geq N$. Using Lemma 1.1, there exists $c > 0$ such that

$$c\sum_{i=1}^n |\alpha_i^m - \alpha_i^r| \leq \left\| \sum_{i=1}^n (\alpha_i^m - \alpha_i^r)\mathbf{x}_i \right\| = \|\mathbf{x}^m - \mathbf{x}^r\| < \epsilon$$

   for all $m, r \geq N$. Dividing by $c > 0$ we get for every i

$$|\alpha_i^m - \alpha_i^r| \leq \sum_{i=1}^n |\alpha_i^m - \alpha_i^r| < \frac{\epsilon}{c}$$

   Thus each sequence α_i^m is a Cauchy sequence in $\Re$. Since $\Re$ is complete, each sequence converges to some limit $\alpha_i \in \Re$.

2. Let

$$\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \cdots + \alpha_n\mathbf{x}_n$$

   Then $\mathbf{x} \in X$ and

$$\|\mathbf{x}^m - \mathbf{x}\| = \left\| \sum_{i=1}^n (\alpha_i^m - \alpha_i)\mathbf{x}_i \right\| \leq \sum_{i=1}^n |\alpha_i^m - \alpha_i| \, \|\mathbf{x}_i\|$$

   Since $\alpha_i^m \to \alpha_i$ for every i, $\|\mathbf{x}^m - \mathbf{x}\| \to 0$ which implies that $\mathbf{x}^m \to \mathbf{x}$.

3. Since $(\mathbf{x}^m)$ was an arbitrary Cauchy sequence, we have shown that every Cauchy sequence in X converges. Hence X is complete.

1.212 Let S be an open set according to the $\|\cdot\|_a$ and let $\mathbf{x}_0$ be a point in S. Since S is open, it contains an open ball in the $\|\cdot\|_a$ topology about $\mathbf{x}_0$, namely $B_a(\mathbf{x}_0, r) = \{\, \mathbf{x} \in X : \|\mathbf{x} - \mathbf{x}_0\|_a < r \,\} \subseteq S$ Let

$$B_b(\mathbf{x}_0, r) = \{\, \mathbf{x} \in X : \|\mathbf{x} - \mathbf{x}_0\|_b < r \,\}$$

be the open ball about $\mathbf{x}_0$ in the $\|\cdot\|_b$ topology. The condition (1.15) implies that $B_b(\mathbf{x}_0, r) \subseteq B_a(\mathbf{x}_0, r) \subseteq S$ and therefore

$$\mathbf{x}_0 \in B_b(\mathbf{x}_0, r) \subset S$$

S is open in the $\|\cdot\|_b$ topology. Similarly, any S open in the $\|\cdot\|_b$ topology is open in the $\|\cdot\|_a$ topology.

1.213 Let X be a normed linear space of dimension n. and let $\{\, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \,\}$ be a basis for X. Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on X. Every $\mathbf{x} \in X$ has a unique representation

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_n \mathbf{x}_n$$

Repeated application of the triangle inequality gives

$$\begin{aligned} \|\mathbf{x}\|_a &= \|\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_n \mathbf{x}_n\|_a \\ &\leq \sum_{i=1}^n \|\alpha_i \mathbf{x}_i\|_a \\ &= \sum_{i=1}^n |\alpha_i| \, \|\mathbf{x}_i\|_a \\ &\leq k \sum_{i=1}^n |\alpha_i| \end{aligned}$$

where $k = \max_i \|\mathbf{x}_i\|$.

By Lemma 1.1, there is a positive constant c such that

$$\sum_{i=1}^n |\alpha_i| \leq \|\mathbf{x}\|_b / c$$

Combining these two inequalities, we have

$$\|\mathbf{x}\|_a \leq k \, \|\mathbf{x}\|_b / c$$

Setting $A = c/k > 0$, we have shown

$$A \, \|\mathbf{x}\|_a \leq \|\mathbf{x}\|_b$$

The other inequality in (1.15) is obtained by interchanging the roles of $\|\cdot\|_a$ and $\|\cdot\|_b$.

1.214 Assume $\mathbf{x}^n \to \mathbf{x} = (x_1, x_2, \dots, x_n)$. Then, for every $\epsilon > 0$, there exists some N such that $\|\mathbf{x}^n - \mathbf{x}\|_\infty < \epsilon$. Therefore, for $i = 1, 2, \dots, n$

$$|x_i^n - x_i| \leq \max_i |x_i^n - x_i| = \|\mathbf{x}^n - \mathbf{x}\|_\infty < \epsilon$$

Therefore $x_i^n \to \mathbf{x}_i$.

Conversely, assume that $(\mathbf{x}^n)$ is a sequence in $\Re^n$ with $x_i^n \to x_i$ for every coordinate i. Choose some $\epsilon > 0$. For every i, there exists some integer N_i such that

$$|x_i^n - x_i| < \epsilon \text{ for every } \ n \geq N_i$$

Let $N = \max_i\{ N_1, N_2, \dots, N_n \}$. Then

$$|x_i^n - x_i| < \epsilon \text{ for every } \ n \geq N$$

and

$$\|\mathbf{x}^n - \mathbf{x}\|_\infty = \max_i |x_i^n - x_i| < \epsilon \text{ for every } \ n \geq N$$

That is, $\mathbf{x}^n \to \mathbf{x}$.

A similar proof can be given using the Euclidean norm $\|\cdot\|_2$, but it is slightly more complicated. This illustrates an instance where the sup norm is more tractable.

1.215
1. Let $S \subseteq X$ be closed and bounded and let $\mathbf{x}^m$ be a sequence in S. Every term $\mathbf{x}^m$ has a representation

$$\mathbf{x}^m = \sum_{i=1}^n \alpha_i^m \mathbf{x}_i$$

Since S is bounded, so is $\mathbf{x}^m$. That is, there exists k such that $\|\mathbf{x}^m\| \leq k$ for all m. Applying Lemma 1.1, there is a positive constant c such that

$$c \sum_{i=1}^n |\alpha_i| \leq \|\mathbf{x}^m\| \leq k$$

Hence, for every i, the sequence of scalars α_i^n is bounded.

2. By the Bolzano-Weierstrass theorem (Exercise 1.119), the sequence α_1^m has a convergent subsequence with limit α_1. Let $x_{(1)}^m$ be the corresponding subsequence of $\mathbf{x}^m$.
3. Similarly, $x_{(1)}^m$ has a subsequence for which the corresponding scalars α_2^m converge to α^2. Repeating this process n times (this is were finiteness is important), we deduce the existence of a subsequence $x_{(n)}^m$ whose scalars converge to $(\alpha_1, \alpha_2, \dots, \alpha_n)$.
4. Let

$$\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{x}_i$$

Since $\alpha_i^m \to \alpha_i$ for every i, $\|\mathbf{x}^m - \mathbf{x}\| \to 0$ which implies that $\mathbf{x}^m \to \mathbf{x}$.

5. Since S is closed, $\mathbf{x} \in S$.
6. We have shown that every sequence in S has a subsequence which converges in S. S is compact.

1.216 Let $\mathbf{x}$ and $\mathbf{y}$ belong to $B = \{ \mathbf{x} : \|\mathbf{x}\| < 1 \}$, the unit ball in the normed linear space X. Then $\|\mathbf{x}\|, \|\mathbf{y}\| < 1$. By the triangle inequality

$$\|\alpha\mathbf{x} + (1-\alpha)\mathbf{y}\| \leq \alpha \|\mathbf{x}\| + (1-\alpha)\|\mathbf{y}\| \leq \alpha + (1-\alpha) = 1$$

Hence $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in B$.

1.217 If int S is empty, it is trivially convex. Therefore, assume int $S \neq \emptyset$ and let $\mathbf{x}, \mathbf{y} \in \text{int } S$. We must show that $\mathbf{z} = \alpha \mathbf{x} + (1-\alpha)\mathbf{y} \in \text{int } S$.

Since $\mathbf{x}, \mathbf{y} \in \text{int } S$, there exists some $r > 0$ such that the open balls $B(\mathbf{x}, r)$ and $B(\mathbf{y}, r)$ are both contained in int S. Let $\mathbf{w}$ be any vector with $\|\mathbf{w}\| < r$. The point

$$\mathbf{z} + \mathbf{w} = \alpha(\mathbf{x} + \mathbf{w}) + (1-\alpha)(\mathbf{y} + \mathbf{w}) \in S$$

since $\mathbf{x} + \mathbf{w} \in B(\mathbf{x}, r) \subset S$ and $\mathbf{y} + \mathbf{w} \in B(\mathbf{y}, r) \subset S$ and S is convex. Hence $\mathbf{z}$ is an interior point of S.

Similarly, if $\overline{S}$ is empty, it is trivially convex. Therefore, assume $\overline{S} \neq \emptyset$ and let $\mathbf{x}, \mathbf{y} \in \overline{S}$. Choose some α. We must show that $z = \alpha \mathbf{x} + (1-\alpha)\mathbf{y} \in \overline{S}$.

There exist sequences $(\mathbf{x}^n)$ and $(\mathbf{y}^n)$ in S which converge to $\mathbf{x}$ and $\mathbf{y}$ respectively (Exercise 1.105). Since S is convex, the sequence $(\alpha \mathbf{x}^n + (1-\alpha)\mathbf{y}^n)$ lies in S and moreover converges to $\alpha \mathbf{x} + (1-\alpha)\mathbf{y} = \mathbf{z}$ (Exercise 1.202). Therefore z is the limit of a sequence in S and hence $z \in \overline{S}$. Therefore, $\overline{S}$ is convex.

1.218 Let $\bar{\mathbf{x}} = \alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$ for some $\alpha \in (0,1)$. Since $\mathbf{x}_1 \in \overline{S}$,

$$\begin{aligned} \mathbf{x}_1 &\in S + rB \\ \alpha \mathbf{x}_1 &\in \alpha(S + rB) \end{aligned}$$

The open ball about $\bar{\mathbf{x}}$ of radius r is

$$\begin{aligned} B(\bar{\mathbf{x}}, r) &= \bar{\mathbf{x}} + rB \\ &= \alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 + rB \\ &\subseteq \alpha(S + rB) + (1-\alpha)\mathbf{x}_2 + rB \\ &= \alpha S + (1-\alpha)\mathbf{x}_2 + (1+\alpha)rB \\ &= \alpha S + (1-\alpha)\left(\mathbf{x}_2 + \frac{1+\alpha}{1-\alpha} rB\right) \end{aligned}$$

Since $\mathbf{x}_2 \in \text{int } S$

$$\mathbf{x}_2 + \frac{1+\alpha}{1-\alpha} rB = B\left(\mathbf{x}_2, \frac{1+\alpha}{1-\alpha} r\right) \subseteq S$$

for sufficiently small r. For such r

$$\begin{aligned} B(\bar{\mathbf{x}}, r) &\subseteq \alpha S + (1-\alpha)S \\ &= S \end{aligned}$$

by Exercise 1.168. Therefore $\bar{\mathbf{x}} \in \text{int } S$.

1.219 It is easy to show that

$$\overline{S} \subseteq \bigcap_{i \in I} \overline{S_i}$$

To show the converse, choose any $\mathbf{x} \in S$ and let $\mathbf{x}_0 \in \overline{S_i}$ for every $i \in I$. By Exercise 1.218, $\alpha \mathbf{x} + (1-\alpha)\mathbf{x}_0 \in S_i$ for all $0 < \alpha < 1$. This implies that $\alpha \mathbf{x} + (1-\alpha)\mathbf{x}_0 \in \cap_{i \in I} S_i = S$ for all $0 < \alpha < 1$, and therefore that $\mathbf{x}_0 = \lim_{\alpha \to 0} \alpha \mathbf{x} + (1-\alpha)\mathbf{x}_0 \in \overline{S}$.

1.220 Assume that $\mathbf{x} \in \text{int } S$. Then, there exists some r such that

$$B(\mathbf{x}, r) = \mathbf{x} + rB \subseteq S$$

Let $\mathbf{y}$ be any element in the unit ball B. Then $-\mathbf{y} \in B$ and

$$\mathbf{x}_1 = \mathbf{x} + r\mathbf{y} \in S$$
$$\mathbf{x}_2 = \mathbf{x} - r\mathbf{y} \in S$$

so that

$$\mathbf{x} = \frac{1}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_2$$

$\mathbf{x}$ is not an extreme point. We have shown that no interior point is an extreme point; hence every extreme point must be a boundary point.

1.221 We showed in Exercise 1.220 that $\text{ext}(S) \subseteq \text{b}(S)$. To show the converse, assume that $\mathbf{x}$ is a boundary point which is not an extreme point. That is, there exist $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that

$$\mathbf{x} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2 \qquad 0 < \alpha < 1$$

This contradicts the assumption that S is strictly convex.

1.222 If S is open, int $S = S$. Since S is convex

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S = \text{int } S \text{ for every } \; 0 \leq \alpha \leq 1$$

A fortiori for every $\mathbf{x} \neq \mathbf{y}$

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in S = \text{int } S \text{ for every } \; 0 < \alpha < 1$$

S is strictly convex.

1.223 Let S be open and $\mathbf{x} \in \text{conv } S$. That is

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$$

with $\mathbf{x}_i \in S$, $\alpha_i \in [0,1]$ and $\sum_i \alpha_i = 1$. The open ball about $\mathbf{x}$

$$\begin{aligned} B(\mathbf{x}, r) &= \mathbf{x} + rB \\ &= \left(\sum_{i=1}^{n} \alpha_i \mathbf{x}_i\right) + rB \\ &= \left(\sum_{i=1}^{n} \alpha_i \mathbf{x}_i\right) + \left(\sum_{i=1}^{n} \alpha_i rB\right) \\ &= \sum_{i=1}^{n} (\alpha_i \mathbf{x}_i + rB) \end{aligned}$$

Since S is open, there exists some r such that $\mathbf{x}_i + rB \in S$ for all i. For this r

$$B(\mathbf{x}, r) \subseteq \text{conv } S$$

Therefore conv S is open.

1.224

$$\text{conv } S = \{\, (x_1, x_2) \in \Re^2 : x_2 > 0 \,\}$$

1.225 S is closed and bounded (Proposition 1.1).

1. S is bounded, that is there exists some K such that $\|\mathbf{x}\| < K$ for every $\mathbf{x} \in S$. Let $\mathbf{x} \in \operatorname{conv} S$. $\mathbf{x}$ is a convex combination of a finite number of points in S, that is

$$\mathbf{x} = \sum_{i=1}^{m} \alpha_i \mathbf{x}_i$$

with $x_i \in S$, $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i = 1$. By the triangle inequality

$$\|\mathbf{x}\| \leq \sum_{i=1}^{m} \alpha_i \|\mathbf{x}_i\| < K$$

Therefore conv S is bounded.

2. Let $\mathbf{x}$ belong to $\overline{\operatorname{conv} S}$. Then, there exists a sequence $(\mathbf{x}^k)$ in conv S which converges to $\mathbf{x}$. By Carathéodory's theorem, each term $\mathbf{x}^k$ is a convex combination of at most $n+1$ points, that is

$$\mathbf{x}^k = \sum_{i=1}^{n+1} \alpha_i^k \mathbf{x}_i^k$$

where $\mathbf{x}_i^k \in S$.

For each i, the sequence $(\mathbf{x}_i^k)$ lies in a compact set S and hence contains a convergent subsequence. Similarly, the sequence of coefficients $(\alpha_i^k) \in [0,1]$ is bounded and contains a convergent subsequence (Bolzano-Weierstrass theorem, Exercise 1.119). Proceeding coordinate by coordinate as in Exercise 1.215, we can construct convergent subsequences $\alpha_i^k \to \alpha_i$ and $\mathbf{x}_i^k - \mathbf{x}_i$.

3. Let

$$\mathbf{x} = \sum_{i=1}^{n+1} \alpha_i \mathbf{x}_i$$

Since

$$\begin{aligned}
\|\mathbf{x}^k - \mathbf{x}\| &= \left\| \sum_{i=1}^{n+1} (\alpha_i^k \mathbf{x}_i^k - \alpha_i \mathbf{x}_i) \right\| \\
&\leq \sum_{i=1}^{n+1} \left\| \alpha_i^k \mathbf{x}_i^k - \alpha_i \mathbf{x}_i \right\| \\
&= \sum_{i=1}^{n+1} \left\| \alpha_i^k \mathbf{x}_i^k - \alpha_i \mathbf{x}_i^k + \alpha_i \mathbf{x}_i^k - \alpha_i \mathbf{x}_i \right\| \\
&= \sum_{i=1}^{n+1} \left| \alpha_i^k - \alpha_i \right| \left\| \mathbf{x}_i^k \right\| + \sum_{i=1}^{n+1} \left| \alpha_i \right| \left\| \mathbf{x}_i^k - \mathbf{x}_i \right\| \\
&\to 0
\end{aligned}$$

as $\alpha_i^k \to \alpha_i$, $\mathbf{x}_i^k - \mathbf{x}_i$, α_i and $\mathbf{x}^k$ are bounded. Therefore $\mathbf{x}^k \to \mathbf{x}$.

4. Since $\alpha_i^k \geq 0$ and $\sum_{i=1}^{n+1} \alpha_i^k = 1$ for every k, we conclude that $\alpha_i = \lim \alpha_i^k \geq 0$ and $\sum_{i=1}^{n+1} \alpha_i = 1$. Furthermore, since S is closed, $\mathbf{x}_i \in S$ for every i and therefore $\mathbf{x} = \sum_{i=1}^{n+1} \alpha_i \mathbf{x}_i \in \operatorname{conv} S$.

5. We have shown that $\overline{\text{conv } S} \subseteq \text{conv } S$, that is conv S is closed.
6. conv S is a closed and bounded subset of a finite dimensional space, and hence conv S is compact (Proposition 1.4 and Exercise 1.215).

1.226 1. S is bounded. Therefore, there exists some c such that $\|\mathbf{x}\|_\infty = \max_i |x_i| < c$ for every $\mathbf{x} \in S$. That is $-c \leq x_i \leq c$ so that

$$\mathbf{x} \in C = \{\, \mathbf{x} = (x_1, x_2, \dots, x_n) \in \Re^n : -c \leq x_i \leq c \text{ for every } \; i \,\}$$

Therefore $S \subset K$.

2. Exercise 1.177.
3. C is the convex hull of a finite set and hence is compact (Exercise 1.225)
4. S is a closed subset of a compact set and hence is compact (Exercise 1.110).

1.227 A polytope is the convex hull of a finite set. Any finite set is compact.

1.228 The unit simplex Δ^{n-1} in $\Re^n$ is the convex hull of the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$, that is

$$\begin{aligned}\Delta^{n-1} &= \text{conv } \{\, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \,\} \\ &= \Big\{ \, (x_1, x_2, \dots, x_n) \in \Re^n : x_i \geq 0 \text{ and } \sum x_i = 1 \Big\}\end{aligned}$$

This simplex has a nonempty relative interior, namely

$$\text{ri } S = \Big\{ \, (x_1, x_2, \dots, x_n) \in \Re^n : x_i > 0 \text{ and } \sum x_i < 1 \Big\}$$

1.229 Let $n = \dim S$. By Exercise 1.182, S contains a simplex S^n of the same dimension. That is, there exist n vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ such that

$$\begin{aligned}S^n &= \text{conv } \{\, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \,\} \\ &= \{\, \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n : \\ &\alpha_1, \alpha_2, \dots, \alpha_n \geq 0, \\ &\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1 \,\}\end{aligned}$$

Analogous to the previous part, the relative interior of S^n is

$$\begin{aligned}\text{ri } S^n &= \text{conv } \{\, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \,\} \\ &= \{\, \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n : \\ &\alpha_1, \alpha_2, \dots, \alpha_n > 0, \\ &\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1 \,\}\end{aligned}$$

which is nonempty.

Note, the proposition is trivially true for a set containing a single point ($n = 0$), since this point is the whole affine space.

1.230 If int $S \neq \emptyset$, then aff $S = X$ and ri $S =$ int S. The converse follows from Exercise 1.229.

1.231 Since

$$m > \inf_{\mathbf{x} \in X} \sum_{i=1}^{n} p_i x_i$$

there exists some $\mathbf{x} \in X$ such that

$$\sum_{i=1}^{n} p_i x_i \leq m$$

Therefore $\mathbf{x} \in X(\mathbf{p}, m)$ which is nonempty.

Let $\check{p} = \min_i p_i$ be the lowest price of the n goods. Then $X(\mathbf{p}, m) \subseteq B(\mathbf{0}, m/\check{p})$ and so is bounded. (That is, no component of an affordable bundle can contain more than $m/\check{p}$ units.)

To show that $X(\mathbf{p}, m)$ is closed, let $(\mathbf{x}^n)$ be a sequence of consumption bundles in $X(\mathbf{p}, m)$. Since $X(\mathbf{p}, m)$ is bounded, $\mathbf{x}^n \to \mathbf{x} \in X$. Furthermore

$$p_1 x_1^n + p_2 x_2^n + \cdots + p_n x_n^n \leq m \text{ for every } \ n$$

This implies that

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \leq m$$

so that $\mathbf{x}^n \to \mathbf{x} \in X(\mathbf{p}, m)$. Therefore $X(\mathbf{p}, m)$ is closed.

We have shown that $X(\mathbf{p}, m)$ is a closed and bounded subset of $\Re^n$. Hence it is compact (Proposition 1.4).

1.232 Let $\mathbf{x}, \mathbf{y} \in X(\mathbf{p}, m)$. That is

$$\sum_{i=1}^{n} p_i x_i \leq m$$
$$\sum_{i=1}^{n} p_i y_i \leq m$$

For any $\alpha \in [0, 1]$, the cost of the weighted average bundle $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$ (where each component $z_i = \alpha x_i + (1-\alpha) y_i$) is

$$\begin{aligned}
\sum_{i=1}^{n} p_i z_i &= \sum_{i=1}^{n} p_i (\alpha x_i + (1-\alpha) y_i \\
&= \alpha \sum_{i=1}^{n} p_i x_i + (1-\alpha) \sum_{i=1}^{n} p_i y_i \\
&\leq \alpha m + (1-\alpha) m \\
&= m
\end{aligned}$$

Therefore $\mathbf{z} \in X(\mathbf{p}, m)$. The budget set $X(\mathbf{p}, m)$ is convex.

1.233 1. Assume that $\succ$ is strongly monotone. Let $\mathbf{x}, \mathbf{y} \in X$ with $\mathbf{x} \geq \mathbf{y}$.

Either $\mathbf{x} \gneq \mathbf{y}$ so that $\mathbf{x} \succ \mathbf{y}$ by strong monotonicity

or $\mathbf{x} = \mathbf{y}$ so that $\mathbf{x} \succsim \mathbf{y}$ by reflexivity.

In either case, $\mathbf{x} \succsim \mathbf{y}$ so that $\succsim$ is weakly monotonic.

2. Again, assume that $\succsim$ is strongly monotonic and let $\mathbf{y} \in X$. X is open (relative to itself). Therefore, there exists some r such that

$$B(\mathbf{y}, r) = \mathbf{y} + rB \subseteq X$$

Let $\mathbf{x} = \mathbf{y} + r\mathbf{e}_1$ be the consumption bundle containing r more units of good 1. Then $\mathbf{e}_1 \in B$, $\mathbf{x} \in B(\mathbf{y}, r)$ and therefore $\|\mathbf{x} - \mathbf{y}\| < r$. Furthermore, $\mathbf{x} \gneq \mathbf{y}$ and therefore $\mathbf{x} \succ \mathbf{y}$.

3. Assume $\succsim$ is locally nonsatiated. Then, for every $\mathbf{x} \in X$, there exists some $\mathbf{y} \in X$ such that $\mathbf{y} \succ \mathbf{x}$. Therefore, there is no best element.

1.234 Assume otherwise, that is assume that $\mathbf{x}^* \succsim \mathbf{x}$ for every $\mathbf{x} \in B(\mathbf{p}, m)$ but that $\sum_{i=1}^n p_i x_i < m$. Let $r = m - \sum_{i=1}^n p_i x_i$ be the unspent income. Spending the residual on good 1, the commodity bundle $\mathbf{x} = \mathbf{x}^* + \frac{r}{p_1}\mathbf{e}_1$ is affordable

$$\sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_i x_i^* + p_1 \frac{r}{p_1} = m$$

Moreover, since $\mathbf{x} \gneq \mathbf{x}^*$, $\mathbf{x} \succ \mathbf{x}^*$, which contradicts the assumption that $\mathbf{x}^*$ is the best element in $X(\mathbf{p}, m)$.

1.235 Assume otherwise, that is assume that $\mathbf{x}^* \succsim \mathbf{x}$ for every $\mathbf{x} \in B(\mathbf{p}, m)$ but that $\sum_{i=1}^n p_i x_i^* < m$. This implies that $\mathbf{x}^* \in \operatorname{int} X(\mathbf{p}, m)$. Therefore, there exists a neighborhood N of $\mathbf{x}^*$ with $N \subseteq X(\mathbf{p}, m)$. Within this neighborhood, there exists some $\mathbf{x} \in N \subseteq X(\mathbf{p}, m)$ with $\mathbf{x} \succ \mathbf{x}^*$, which contradicts the assumption that $\mathbf{x}^*$ is the best element in $X(\mathbf{p}, m)$.

1.236

1. Assume $\succsim$ is continuous. Choose some $\mathbf{y} \in X$. For any $\mathbf{x}_0$ in $\succ(\mathbf{y})$, $\mathbf{x}_0 \succ \mathbf{y}$ and (since $\succsim$ is continuous) there exists some neighborhood $S(\mathbf{x}_0)$ such that $\mathbf{x} \succ \mathbf{y}$ for every $\mathbf{x} \in S(\mathbf{x}_0)$. That is, $S(\mathbf{x}_0) \subseteq \succ(\mathbf{y})$ and $\succ(\mathbf{y})$ is open.

   Similarly, for any $\mathbf{x}_0 \in \prec(\mathbf{y})$, $\mathbf{x}_0 \prec \mathbf{y}$ and there exists some neighborhood $S(\mathbf{x}_0)$ such that $\mathbf{x} \prec \mathbf{y}$ for every $\mathbf{x} \in S(\mathbf{x}_0)$. Thus $S(\mathbf{x}_0) \subseteq \prec(\mathbf{y})$ and $\prec(\mathbf{y})$ is open.

2. Conversely, assume that the sets $\succ(\mathbf{y}) = \{\, \mathbf{x} : \mathbf{x} \succ \mathbf{y} \,\}$ and $\prec(\mathbf{y}) = \{\, \mathbf{x} : \mathbf{x} \prec \mathbf{y} \,\}$ are open in $\mathbf{x}$. Assume $\mathbf{x}_0 \succ \mathbf{y}_0$.
   (a) Suppose there exists some $\mathbf{y}$ such that $\mathbf{x}_0 \succ \mathbf{y} \succ \mathbf{z}_0$. Then $\mathbf{x}_0 \in \succ(\mathbf{y})$, which is open by assumption. That is, $\succ(\mathbf{y})$ is an open neighborhood of $\mathbf{x}_0$ and $\mathbf{x} \succ \mathbf{y}$ for every $\mathbf{x} \in \succ(\mathbf{y})$. Similarly, $\prec(\mathbf{y})$ is an open neighborhood of $\mathbf{z}_0$ for which $\mathbf{y} \succ \mathbf{z}$ for every $\mathbf{z} \in \prec(\mathbf{y})$. Therefore $S(\mathbf{x}_0) = \succ(\mathbf{y})$ and $S(\mathbf{z}_0) = \prec(\mathbf{y})$ are the required neighborhoods of $\mathbf{x}_0$ and $\mathbf{z}_0$ respectively such that

   $$\mathbf{x} \succ \mathbf{y} \succ \mathbf{z} \qquad \text{for every } \mathbf{x} \in S(\mathbf{x}_0) \text{ and } \mathbf{y} \in S(\mathbf{z}_0)$$

   (b) Suppose there is no $\mathbf{y}$ such that $\mathbf{x}_0 \succ \mathbf{y} \succ \mathbf{z}_0$.
      i. By assumption
         - $\succ(\mathbf{z}_0)$ is open
         - $\mathbf{x}_0 \succ \mathbf{z}_0$ which implies $\mathbf{x}_0 \in \succ(\mathbf{z}_0)$,

         Therefore $\succ(\mathbf{z}_0)$ is an open neighborhood of $\mathbf{x}_0$.
      ii. Since $\succsim$ is complete, either $\mathbf{y} \prec \mathbf{x}_0$ or $\mathbf{y} \succsim \mathbf{x}_0$ for every $\mathbf{y} \in X$ (Exercise 1.56. Since there is no $\mathbf{y}$ such that $\mathbf{x}_0 \succ \mathbf{y} \succ \mathbf{z}_0$

         $$\mathbf{y} \succ \mathbf{z}_0 \implies \mathbf{y} \not\prec \mathbf{x}_0 \implies \mathbf{y} \succsim \mathbf{x}_0$$

         Therefore $\succ(\mathbf{z}_0) = \succsim(\mathbf{x}_0)$.
      iii. Since $\mathbf{x} \succsim \mathbf{x}_0 \succ \mathbf{z}_0$ for every $\mathbf{x} \in \succsim(\mathbf{x}_0) = \succ(\mathbf{z}_0)$

         $$\mathbf{x} \succ \mathbf{z}_0 \text{ for every } \mathbf{x} \in \succ(\mathbf{z}_0)$$

iv. Therefore $S(\mathbf{x}_0) = \succ(\mathbf{z}_0)$ is an open neighborhood of $\mathbf{x}_0$ such that

$$\mathbf{x} \succ \mathbf{z}_0 \text{ for every } \mathbf{x} \in S(\mathbf{x}_0)$$

Similarly, $S(\mathbf{z}_0) = \prec(\mathbf{x}_0)$ is an open neighborhood of $\mathbf{z}_0$ such that $\mathbf{z} \prec \mathbf{x}_0$ for every $\mathbf{z} \in S(\mathbf{z}_0)$. Consequently

$$\mathbf{x} \succ \mathbf{z} \qquad \text{for every } \mathbf{x} \in S(\mathbf{x}_0) \text{ and } \mathbf{z} \in S(\mathbf{z}_0)$$

3. $\succsim(\mathbf{y}) = \left(\prec(\mathbf{y})\right)^c$ (Exercise 1.56). Therefore, $\succsim(\mathbf{y})$ is closed if and only if $\prec(\mathbf{y})$ is open (Exercise 1.80). Similarly, $\precsim(\mathbf{y})$ is closed if and only if $\succ(\mathbf{y})$ is open.

1.237 1. Let $F = \{ (\mathbf{x}, \mathbf{y}) \in X \times X : \mathbf{x} \succsim \mathbf{y} \}$. Let $((\mathbf{x}^n, \mathbf{y}^n))$ be a sequence in F which converges to $(\mathbf{x}, \mathbf{y})$. Since $(\mathbf{x}^n, \mathbf{y}^n) \in F$, $\mathbf{x}^n \succsim \mathbf{y}^n$ for every n. By assumption, $\mathbf{x} \succsim \mathbf{y}$. Therefore, $(\mathbf{x}, \mathbf{y}) \in F$ which establishes that F is closed (Exercise 1.106)

Conversely, assume that F is closed and let $((\mathbf{x}^n, \mathbf{y}^n))$ be a sequence converging to $(\mathbf{x}, \mathbf{y})$ with $\mathbf{x}^n \succsim \mathbf{y}^n$ for every n. Then $((\mathbf{x}^n, \mathbf{y}^n)) \in F$ which implies that $(\mathbf{x}, \mathbf{y}) \in F$. Therefore $\mathbf{x} \succsim \mathbf{y}$.

2. Yes. Setting $\mathbf{y}^n = \mathbf{y}$ for every n, their definition implies that for every sequence $(\mathbf{x}^n)$ in X with $\mathbf{x}^n \succsim \mathbf{y}$, $\mathbf{x} = \lim \mathbf{x}^n \succsim \mathbf{y}$. That is, the upper contour set $\succsim(\mathbf{y}) = \{ \mathbf{x} : \mathbf{x} \succsim \mathbf{y} \}$ is closed. Similarly, the lower contour set $\precsim(\mathbf{y})$ is closed.

Conversely, assume that the preference relation is continuous (in our definition). We show that the set $G = \{ (\mathbf{x}, \mathbf{y}) : \mathbf{x} \prec \mathbf{y} \}$ is open. Let $(\mathbf{x}^0, \mathbf{y}^0) \in G$. Then $\mathbf{x}^0 \prec \mathbf{y}^0$. By continuity, there exists neighborhoods $S(\mathbf{x}_0)$ and $S(\mathbf{y}_0)$ of $\mathbf{x}_0$ and $\mathbf{y}_0$ such that $\mathbf{x} \prec \mathbf{y}$ for every $\mathbf{x} \in S(\mathbf{x}_0)$ and $\mathbf{y} \in S(\mathbf{y}_0)$. Hence, for every $(\mathbf{x}, \mathbf{y}) \in N = S(\mathbf{x}_0) \times S(\mathbf{y}_0)$, $\mathbf{x} \prec \mathbf{y}$. Therefore $N \subseteq G$ which implies that G is open. Consequently $G^c = \{ (\mathbf{x}, \mathbf{y}) : \mathbf{x} \succsim \mathbf{y} \}$ is closed.

1.238 Assume the contrary. That is, assume there is no $\mathbf{y}$ with $\mathbf{x} \succ \mathbf{y} \succ \mathbf{z}$. Since $\succsim$ is complete, either $\mathbf{y} \prec \mathbf{x}_0$ or $\mathbf{y} \succsim \mathbf{x}_0$ for every $\mathbf{y} \in X$ (Exercise 1.56). Since there is no $\mathbf{y}$ such that $\mathbf{x}_0 \succ \mathbf{y} \succ \mathbf{z}_0$

$$\mathbf{y} \succ \mathbf{z}_0 \implies \mathbf{y} \not\prec \mathbf{x}_0 \implies \mathbf{y} \succsim \mathbf{x}_0$$

Therefore $\succ(\mathbf{z}_0) = \succsim(\mathbf{x}_0)$. By continuity, $\succ(\mathbf{z}_0)$ is open and $\succsim(\mathbf{x}_0)$ is closed. Hence $\succ(\mathbf{z}_0) = \succsim(\mathbf{x}_0)$ is both open and closed (Exercise 1.83).

Alternatively, $\succsim(\mathbf{x}_0)$ and $\precsim(\mathbf{z}_0)$ are both open sets which partition X. This contradicts the assumption that X is connected.

1.239 Let X^* denote the set of best elements. As demonstrated in the preceding proof

$$X^* = \bigcap_{\mathbf{y} \in X} \succsim(\mathbf{y}_i)$$

Therefore X^* is closed (Exercise 1.85) and hence compact (Exercise 1.110).

1.240 Assume for simplicity that $p_1 = p_2 = 1$ and that $m = 1$. Then, the budget set is

$$B(1,1) = \{ \mathbf{x} \in \Re^2_{++} : x_1 + x_2 \leq 1 \}$$

The consumer would like to spend as much as possible of her income on good 1. However, the point $(1,0)$ is not feasible, since $(1,0) \notin X$.

1.241 Essentially, consumer theory (in economics) is concerned with predicting the way in which consumer purchases vary with changes in observable parameters such as prices and incomes. Predictions are deduced by assuming that the consumer will consistently choose the best affordable alternative in her budget set. The theory would be empty if there was no such optimal choice.

1.242 1. Let $X^0 = X \cap \Re^n_+$. Then X^0 is compact and $X^1 \subseteq X^0$. Define the order $\mathbf{x} \succsim_1 \mathbf{y}$ if and only if $d_1(\mathbf{x}) \leq d_1(\mathbf{y})$. Then $\succsim_1$ is continuous on X and

$$X^1 = \{\, \mathbf{x} \in X : d_1(\mathbf{x}) \leq d_1(\mathbf{y}) \text{ for every } \ \mathbf{y} \in X \,\}$$

is the set of best elements in X with respect to the order $\succsim_1$. By Exercise 1.239, X^1 is nonempty and compact.

2. Assume X^{k-1} is compact. Define the order $\mathbf{x} \succsim_k \mathbf{y}$ if and only if $d_k(\mathbf{x}) \leq d_k(\mathbf{y})$. Then $\succsim_k$ is continuous on X^{k-1} and

$$X^k = \{\, \mathbf{x} \in X^{k-1} : d_k(\mathbf{x}) \leq d_k(\mathbf{y}) \text{ for every } \ \mathbf{y} \in X^{k-1} \,\}$$

is the set of best elements in X^{k-1} with respect to the order $\succsim_k$. By Exercise 1.239, X^k is nonempty and compact.

3. Assume $\mathbf{x} \in \text{Nu}$. Then

$$\mathbf{x} \succsim \mathbf{y} \text{ for every } \ \mathbf{y} \in X$$
$$\mathbf{d}(\mathbf{x}) \precsim_L \mathbf{d}(\mathbf{y}) \text{ for every } \ \mathbf{y} \in X$$

For every $k = 1, 2, \dots, 2^n$

$$\mathbf{d}_k(\mathbf{x}) \leq \mathbf{d}_k(\mathbf{y}) \text{ for every } \ \mathbf{y} \in X$$

which implies $\mathbf{x} \in X^k$. In particular $\mathbf{x} \in X^{2^n}$. Therefore $\text{Nu} \subseteq X^{2^n}$.

Suppose $\text{Nu} \subset X^{2^n}$. Then there exists some $\mathbf{x}, \mathbf{y} \in X$, $\mathbf{x} \notin X^{2^n}$ and $\mathbf{y} \in X^{2^n}$ such that $\mathbf{x} \succsim^d \mathbf{y}$. Let k be the smallest integer such that $\mathbf{x} \notin X^k$. Then $d_k(\mathbf{x}) > d_k(\mathbf{y})$. But $\mathbf{x} \in X^l$ for every $l < k$ and therefore $d_l(\mathbf{x}) = d_l(\mathbf{y})$ for $l = 1, 2, \dots, k-1$. This means that $\mathbf{d}(\mathbf{y}) \prec^L \mathbf{d}(\mathbf{x})$ so that $\mathbf{x} \prec^d \mathbf{y}$. This contradiction establishes that $\text{Nu} = X^{2^n}$.

1.243 Assume $\succsim$ is convex. Choose any $\mathbf{y} \in X$ and let $\mathbf{x} \in \succsim(\mathbf{y})$. Then $\mathbf{x} \succsim \mathbf{y}$. Since $\succsim$ is convex, this implies that

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{y} \qquad \text{for every } \ 0 \leq \alpha \leq 1$$

and therefore

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \succsim(\mathbf{y}) \qquad \text{for every } \ 0 \leq \alpha \leq 1$$

Therefore $\succsim(\mathbf{y})$ is convex.

To show the converse, assume that $\succsim(\mathbf{y})$ is convex for every $\mathbf{y} \in X$. Choose $\mathbf{x}, \mathbf{y} \in X$. Interchanging $\mathbf{x}$ and $\mathbf{y}$ if necessary, we can assume that $\mathbf{x} \succsim \mathbf{y}$ so that $\mathbf{x} \in \succsim(\mathbf{y})$. Of course, $\mathbf{y} \in \succsim(\mathbf{y})$. Since $\succsim(\mathbf{y})$ is convex

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \succsim(\mathbf{y}) \qquad \text{for every } \ 0 \leq \alpha \leq 1$$

which implies

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{y} \qquad \text{for every } \ 0 \leq \alpha \leq 1$$

1.244 X^* may be empty, in which case it is trivially convex. Otherwise, let $\mathbf{x}^* \in X^*$. For every $\mathbf{x} \in X^*$

$$\mathbf{x} \succsim \mathbf{x}^* \text{ which implies } \mathbf{x} \in \succsim(\mathbf{x}^*)$$

Therefore $X^* \subseteq \succsim(\mathbf{x}^*)$. Conversely, by transitivity

$$\mathbf{x} \succsim \mathbf{x}^* \succsim \mathbf{y} \text{ for every } \ \mathbf{y} \in X$$

for every $\mathbf{x} \in \succsim(\mathbf{x}^*)$ which implies $\succsim(\mathbf{x}^*) \subseteq X^*$. Therefore, $X^* = \succsim(\mathbf{x}^*)$ which is convex.

1.245 To show that $\succsim^d$ is strictly convex, assume that $\mathbf{x}, \mathbf{y} \in X$ are such $\mathbf{d}(\mathbf{x}) = \mathbf{d}(\mathbf{y})$ with $\mathbf{x} \neq \mathbf{y}$. Suppose

$$\mathbf{d}(\mathbf{x}) = \big(d(S_1, \mathbf{x}), d(S_2, \mathbf{x}) \dots, d(S_{2^n}, \mathbf{x})\big)$$

In the order $S_1, S_2, \dots, S_{2^n}$, let S_k be the first coalition for which $d(S_k, \mathbf{x}) \neq d(S_k, \mathbf{y})$. That is

$$d(S_j, \mathbf{x}) = d(S_j, \mathbf{y}) \text{ for every } \ j < k \tag{1.18}$$

Since $d(S_k, \mathbf{x}) \neq d(S_k, \mathbf{y})$ and $\mathbf{d}(\mathbf{x})$ is listed in descending order, we must have

$$d(S_k, \mathbf{x}) > d(S_k, \mathbf{y}) \tag{1.19}$$

and

$$d(S_k, \mathbf{x}) \geq d(S_j, \mathbf{y}) \text{ for every } \ j > k \tag{1.20}$$

Choose $0 < \alpha < 1$ and let $\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y}$. For any coalition S

$$\begin{aligned}
d(S, \mathbf{z}) &= w(S) - \sum_{i \in S} z_i \\
&= w(S) - \sum_{i \in S} \big(\alpha x_i + (1-\alpha) y_i\big) \\
&= w(S) - \alpha \sum_{i \in S} x_i - (1-\alpha) \sum_{i \in S} y_i \\
&= \alpha \left(w(S) - \sum_{i \in S} x_i \right) + (1-\alpha) \left(w(S) - \sum_{i \in S} y_i \right) \\
&= \alpha d(S, \mathbf{x}) + (1-\alpha) d(S, \mathbf{y})
\end{aligned}$$

Using (1.18) to (1.20), this implies that

$$\begin{aligned}
d(S_j, \mathbf{z}) &= d(S_j, \mathbf{x}), \qquad && j < k \\
d(S_k, \mathbf{z}) &< d(S_k, \mathbf{x}) \\
d(S_k, \mathbf{z}) &\leq d(S_j, \mathbf{x}), && j > k
\end{aligned}$$

for every $0 < \alpha < 1$, Therefore $\mathbf{d}(\mathbf{z}) \prec_L \mathbf{d}(\mathbf{x})$. Thus $\mathbf{z} \succ^d \mathbf{x}$, which establishes that $\succsim$ is strictly convex.

The set of feasible outcomes is convex. Assume $\mathbf{x}, \mathbf{y} \in \text{Nu} \subseteq X$, $\mathbf{x} \neq \mathbf{y}$. Then $\mathbf{d}(\mathbf{x}) = \mathbf{d}(\mathbf{y})$ and

$$\mathbf{z} = \alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succ^d \mathbf{x}$$

for every $0 < \alpha < 1$ which contradicts the assumption that $\mathbf{x} \in \text{Nu}$. We conclude that the nucleolus contains only one element.

1.246 1. (a) Clearly $\prec(\mathbf{x}_0) \subseteq \precsim(\mathbf{x}_0)$ and $\succ(\mathbf{y}_0) \subseteq \succsim(\mathbf{y}_0)$. Consequently $\prec(\mathbf{x}_0) \cup \succ(\mathbf{y}_0) \subseteq \precsim(\mathbf{x}_0) \cup \succsim(\mathbf{y}_0)$. We claim that these sets are in fact equal.

Let $\mathbf{z} \in \precsim(\mathbf{x}_0) \cup \succsim(\mathbf{y}_0)$. Suppose that $\mathbf{z} \in \precsim(\mathbf{x}_0)$ but $\mathbf{z} \notin \prec(\mathbf{x}_0)$. Then $\mathbf{z} \succsim \mathbf{x}_0$. By transitivity, $\mathbf{z} \succsim \mathbf{x}_0 \succ \mathbf{y}_0$ which implies that $\mathbf{z} \in \succ(\mathbf{y}_0)$. Similarly $\mathbf{z} \in \succsim(\mathbf{y}_0) \setminus \succ(\mathbf{y}_0)$ implies $\mathbf{z} \in \prec(\mathbf{x}_0)$. Therefore

$$\prec(\mathbf{x}_0) \cup \succ(\mathbf{y}_0) = \precsim(\mathbf{x}_0) \cup \succsim(\mathbf{y}_0)$$

(b) By continuity, $\prec(\mathbf{x}_0) \cup \succ(\mathbf{y}_0)$ is open and $\precsim(\mathbf{x}_0) \cup \succsim(\mathbf{y}_0) = \prec(\mathbf{x}_0) \cup \succ(\mathbf{y}_0)$ is closed. Further $\mathbf{x}_0 \succ \mathbf{y}_0$ implies that $\mathbf{x}_0 \in \succ(\mathbf{y}_0)$ so that $\prec(\mathbf{x}_0) \cup \succ(\mathbf{y}_0) \neq \emptyset$. We have established that $\prec(\mathbf{x}_0) \cup \succ(\mathbf{y}_0)$ is a nonempty subset of X which is both open and closed. Since X is connected, this implies (Exercise 1.83) that

$$\prec(\mathbf{x}_0) \cup \succ(\mathbf{y}_0) = X$$

2. (a) By definition, $\mathbf{x} \notin \prec(\mathbf{x})$. So $\prec(\mathbf{x}) \cap \prec(\mathbf{y}) = X$ implies $\mathbf{x} \in \succ(\mathbf{y})$, that is $\mathbf{x} \succsim \mathbf{y}$ contradicting the noncomparability of $\mathbf{x}$ and $\mathbf{y}$. Therefore

$$\prec(\mathbf{x}) \cap \prec(\mathbf{y}) \neq X$$

(b) By assumption, there exists at least one pair $\mathbf{x}_0, \mathbf{y}_0$ such that $\mathbf{x}_0 \succ \mathbf{y}_0$. By the previous part

$$\prec(\mathbf{x}_0) \cup \succ(\mathbf{y}_0) = X$$

This implies either $\mathbf{x} \prec \mathbf{x}_0$ or $\mathbf{x} \succ \mathbf{y}_0$. Without loss of generality, assume $\mathbf{x} \succ \mathbf{y}_0$. Again using the previous part, we have

$$\prec(\mathbf{x}) \cup \succ(\mathbf{y}_0) = X$$

Since $\mathbf{x}$ and $\mathbf{y}$ are not comparable, $\mathbf{y} \notin \prec(\mathbf{x})$ which implies that $\mathbf{y} \in \succ(\mathbf{y}_0)$. Therefore $\mathbf{x} \succ \mathbf{y}_0$ and $\mathbf{y} \succ \mathbf{y}_0$ or alternatively

$$\mathbf{y}_0 \in \prec(\mathbf{x}_0) \cap \succ(\mathbf{y}_0) \neq \emptyset$$

(c) Clearly $\prec(\mathbf{x}) \subseteq \precsim(\mathbf{x})$ and $\succ(\mathbf{y}) \subseteq \succsim(\mathbf{y})$. Consequently

$$\prec(\mathbf{x}) \cap \prec(\mathbf{y}) \subseteq \precsim(\mathbf{x}) \cap \precsim(\mathbf{y})$$

Let $\mathbf{z} \in \precsim(\mathbf{x}) \cap \precsim(\mathbf{y})$. That is, $\mathbf{z} \precsim \mathbf{x}$. If $\mathbf{x} \precsim \mathbf{z}$, then transitivity implies $\mathbf{x} \precsim \mathbf{z} \precsim \mathbf{y}$, which contradicts the noncomparability of $\mathbf{x}$ and $\mathbf{y}$. Consequently $\mathbf{x} \not\precsim \mathbf{z}$ which implies $\mathbf{z} \prec \mathbf{x}$ and $\mathbf{z} \in \prec(\mathbf{x})$. Similarly $\mathbf{z} \in \prec(\mathbf{y})$ and therefore

$$\prec(\mathbf{x}) \cap \prec(\mathbf{y}) = \precsim(\mathbf{x}) \cap \precsim(\mathbf{y})$$

3. If $\mathbf{x}$ and $\mathbf{y}$ are noncomparable, $\prec(\mathbf{x}) \cap \prec(\mathbf{y})$ is a nonempty proper subset of X. By continuity $\prec(\mathbf{x}) \cap \prec(\mathbf{y}) = \precsim(\mathbf{x}) \cap \precsim(\mathbf{y})$ is both open and closed which contradicts the assumption that X is connected (Exercise 1.83). We conclude that $\succsim$ must be complete.

1.247 Assume $\mathbf{x} \succ \mathbf{y}$. Then $\mathbf{x} \in \succ(\mathbf{y})$. Since $\succ(\mathbf{y})$ is open, $\mathbf{x} \in \operatorname{int} \succ(\mathbf{y})$. Also $\mathbf{y} \in \overline{\succ(\mathbf{y})}$. By Exercise 1.218, $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in \operatorname{int} \succ(\mathbf{y})$ for every $0 < \alpha < 1$, which implies

$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succ \mathbf{y} \text{ for every } 0 < \alpha < 1$$

1.248 For every $\mathbf{x} \in X$, there exists some $\mathbf{z}$ such that $\mathbf{z} \succ \mathbf{x}$ (Nonsatiation). For any r, choose some $\alpha \in (0, r/\|\mathbf{x} - \mathbf{z}\|)$ and let $\mathbf{y} = \alpha\mathbf{z} + (1-\alpha)\mathbf{x}$. Then

$$\|\mathbf{x} - \mathbf{y}\| = \alpha \|\mathbf{x} - \mathbf{z}\| < r$$

Moreover, since $\succsim$ is strictly convex,

$$\mathbf{y} = \alpha\mathbf{z} + (1-\alpha)\mathbf{x} \succ \mathbf{x}$$

Thus, $\succsim$ is locally nonsatiated.

We have previously shown that local nonsatiation implies nonsatiation (Exercise 1.233). Consequently, these two properties are equivalent for strictly convex preferences.

1.249 Assume that $\mathbf{x}$ is not strongly Pareto efficient. That is, there exist allocation $\mathbf{y}$ such that $\mathbf{y} \succsim_i \mathbf{x}$ for all i and some individual j for which $\mathbf{y} \succ_j \mathbf{x}$. Take $1-t$ percent of j's consumption and distribute it equally to the other participants. By continuity, there exists some t such that $t\mathbf{y} \succ_j \mathbf{x}$. The other agents receive $\mathbf{y}_i + \frac{1-t}{n-1}\mathbf{y}_j$ which, by monotonicity, they strictly prefer to $\mathbf{x}_i$.

1.250 Assume that $(\mathbf{p}^*, \underline{\mathbf{x}^*})$ is a competitive equilibrium of an exchange economy, but that $\underline{\mathbf{x}}^*$ does not belong to the core of the corresponding market game. Then there exists some coalition S and allocation $\underline{\mathbf{y}} \in W(S)$ such that $\mathbf{y}_i \succ_i \mathbf{x}_i^*$ for every $i \in S$. Since $\underline{\mathbf{y}} \in W(S)$, we must have $\sum_{i \in S} \mathbf{y}_i = \sum_{i \in S} \mathbf{w}_i$.

Since $\underline{\mathbf{x}^*}$ is a competitive equilibrium and $\mathbf{y}_i \succ_i \mathbf{x}_i^*$ for every $i \in S$, $\underline{\mathbf{y}}$ must be unaffordable, that is

$$\sum_{j=1}^{l} p_j y_{ij} > \sum_{j=1}^{l} p_j \mathbf{w}_{ij} \text{ for every } \; i \in S$$

and therefore

$$\sum_{i \in S} \sum_{j=1}^{l} p_j y_{ij} > \sum_{i \in S} \sum_{j=1}^{l} p_j \mathbf{w}_{ij}$$

which contradicts the assumption that $y \in W(S)$.

1.251 Combining the previous exercise with Exercise 1.64

$$\mathbf{x}^* \in \text{core} \subseteq \text{Pareto}$$