The Kuhn-Tucker constraint qualification

It was Kuhn and Tucker who alerted practitioners to the necessity of constraint qualification (Example 5.36) and initiated the search for appropriate conditions. They proposed:

Kuhn-Tucker (KTCQ) For every perturbation $\mathbf{dx} \in L$ there exists a feasible differentiable arc $h\colon [0,1] \to X$ with $h(0) = \mathbf{x}^*$ and $h'(0) = \alpha \mathbf{dx}$ for some positive scalar α.

Define the *cone of attainable directions* $A(\mathbf{x}^*)$ by

$$A(\mathbf{x}^*) = \{\mathbf{dx} \in \Re^n : \text{there exists } \alpha \in \Re_+ \text{ and } \; h\colon [0,1] \to X \\ \text{such that } h(0) = x^* \text{ and } h'(0) = \alpha dx\}$$

Then, the Kuhn-Tucker constraint qualification condition is $A(\mathbf{x}^*) = L(\mathbf{x}^*)$.

Proposition 1 (KTCQ) *Suppose that $\mathbf{x}^*$ is a local solution of*

$$\max_{\mathbf{x} \in X} f(\mathbf{x})$$
$$\textit{subject to } g(\mathbf{x}) \leq \mathbf{0}$$

and satisfies the Kuhn-Tucker constraint qualification $A(\mathbf{x}^*) = L(\mathbf{x}^*)$. Then there exist multipliers $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ such that

$$\nabla f(\mathbf{x}^*) = \sum_{j=1}^{m} \lambda_j \nabla g_j(\mathbf{x}^*) \;\; \textit{and} \;\; \lambda_j g_j(\mathbf{x}^*) = 0, \quad j = 1, 2 \dots, m$$

Proof. We first show that $A(\mathbf{x}^*) \subseteq T(\mathbf{x}^*)$. Take any attainable direction $\mathbf{dx} \in A(\mathbf{x}^*)$. Then there exists a feasible differentiable arc $h\colon [0,1] \to X^0$ with $h(0) = \mathbf{x}^*$ and $D_t h(0) = \alpha \mathbf{dx}$ for some positive scalar α. Choose a sequence $\{t_k\} \subset (0,1]$ which converges to 0. Let $\mathbf{x}^k = h(t_k)$. Then $\mathbf{x}^k \in X^0$ for all k and $\lim_{k \to \infty} \mathbf{x}^k = \mathbf{x}^*$. Letting $\alpha^k = 1/(\alpha t_k)$, it follows that $\alpha_k(\mathbf{x}^k - \mathbf{x}^*) \to \mathbf{dx}$. Hence $\mathbf{dx} \in T(\mathbf{x}^*)$.

Since $T(\mathbf{x}^*) \subseteq L(\mathbf{x}^*)$ (Exercise 5.40), the Kuhn-Tucker constraint qualification condition $A(\mathbf{x}^*) = L(\mathbf{x}^*)$ implies the Abadie constraint qualification $T(\mathbf{x}^*) = A(\mathbf{x}^*)$, and hence the necessity of the Kuhn-Tucker conditions (Proposition 5.4). □

Because the KTCQ is difficult to apply in practice, Arrow, Hurwicz and Uzawa (1961) sought alternative conditions which were easier to apply. The condition with which they are credited (AHUCQ) was designed as a vehicle for establishing the validity of the criteria in Theorem 5.4.2.

Bazarra, Goode and Shetty (1972) show that the Kuhn-Tucker constraint qualification $A(\mathbf{x}^*) = L(\mathbf{x}^*)$ can be weakened to $\overline{A}(\mathbf{x}^*) = L(\mathbf{x}^*)$. Hence

$$L^1(\mathbf{x}^*) \subset A(\mathbf{x}^*) \subset T(\mathbf{x}^*) \subset L^1(\mathbf{x}^*)$$

implies

$$\overline{L^1}(\mathbf{x}^*) \subset \overline{A}(\mathbf{x}^*) \subset T(\mathbf{x}^*) \subset L^1(\mathbf{x}^*)$$

Then the AHUCQ $\overline{L^1}(\mathbf{x}^*) = L(\mathbf{x}^*)$ implies the relaxed Kuhn Tucker constraint qualification $\overline{A}(\mathbf{x}^*) = L(\mathbf{x}^*)$.