Chapter 1: Sets and Spaces

Exercise 1.1 (Direct sum) A linear space X is the *direct sum* of two subspaces S_1 and S_2, denoted $X = S_1 \oplus S_2$, if every $\mathbf{x} \in X$ has a unique representation of the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ where $x_1 \in S_1$ and $\mathbf{x}_2 \in S_2$. Show that

$$X = S_1 \oplus S_2 \iff X = S_1 + S_2 \text{ and } S_1 \cap S_2 = \{0\}$$

Chapter 3: Linear Functions

Exercise 3.1 (Orthogonal complement) The orthogonal complement $S^\perp$ of any subset S in an inner product space is a subspace.

Exercise 3.2 (Projection theorem) Let S be a closed subspace of a Hilbert space X and $\mathbf{y}$ a point outside S. There exists a unique point $\mathbf{x}_0 \in S$ which is closest to $\mathbf{y}$, that is

$$\|\mathbf{x}_0 - \mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| \text{ for every } \mathbf{x} \in S$$

Further, $\mathbf{y} - \mathbf{x}_0$ is orthogonal to S, that is $(\mathbf{y} - \mathbf{x}_0)^T\mathbf{x} = 0$ for every $\mathbf{x} \in S$.

[Hint: Use Exercise 3.73.]

Exercise 3.3 A linear space X is the *direct sum* of two subspaces S_1 and S_2, denoted $X = S_1 \oplus S_2$, if every $\mathbf{x} \in X$ has a unique representation of the form $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ where $x_1 \in S_1$ and $\mathbf{x}_2 \in S_2$. If S is a closed subspace of a Hilbert space X, then

$$X = S \oplus S^\perp$$

[Hint: Use supplementary exercise 3.2.]

Exercise 3.4 If S is a subspace of a Euclidean space X

$$\dim X = \dim S + \dim S^\perp$$

Exercise 3.5 Let X be a normed linear space. Show that every $\mathbf{x} \in X$ defines a unique bounded linear functional F_x on X^* with

$$F_{\mathbf{x}}(f) = f(\mathbf{x}) \text{ for every } f \in X^*$$

and norm

$$\|F_{\mathbf{x}}\| = \|\mathbf{x}\|$$

Chapter 5: Optimization

Exercise 5.1 Conjecture: If $f : \Re \to \Re$ has a local maximum at x^* which is not a strict local maximum, then f is constant in some neighbourhood of x. Prove or provide a counterexample. q

Exercise 5.2 Extend Corollary 5.1.2 to f pseudoconcave.

Exercise 5.3 Suppose f is pseudoconcave. Then, it is locally concave at every stationary point.

Exercise 5.4 (Ramsey pricing) Suppose a monopolist produces at constant marginal cost c, with a fixed cost of k. The monopolist sells in two distinct markets with elasticities of demand ϵ_1 and ϵ_2 respectively.

1. Show that profit maximizing prices are characterized by the inverse elasticity rule

$$\frac{p_i - c}{p_i} = \frac{1}{\epsilon_i}$$

2. Suppose a regulator determines the prices in each market to maximize total surplus as measured by the area under the demand curve

$$S_i(p_i) = \int_{p_i}^{\infty} q_i(\tau) d\tau$$

while ensuring that the monopolist breaks even. That is, the regulator's optimization problem is

$$\max_{p_1, p_2} = \sum_{i=1}^{n} S_i(p_i)$$
$$\text{subject to } \sum_{i=1}^{n} (p_i - c) q_i(p_i) - k = 0$$

Show that optimal (Ramsey) prices are given by

$$\frac{p_i - c}{p_i} = \theta \frac{1}{\epsilon_i}$$

where $0 \leq \theta < 1$. Note that

$$D_{p_i} S_i(p_i) = -q_i(p_i)$$

Exercise 5.5 Solve the problem

$$\max_{x_1, x_2} 4x_1 + 2x_1 x_2 + 2x_2 - 2x_1^2 - 2x_2^2$$
$$\text{subject to } x_1 + 2x_2 \leq 2$$
$$x_2^2 - x_1 \leq 1$$

Exercise 5.6 Suppose that a firm has contracted with its union to hire at least l units of labour at rate w_1 per unit. It can also hire non-union labour at $w_2 < w_1$ per hour. Assume that labour is the only input. Union and non-union labour are equally productive, with diminishing marginal product. Output is sold at a fixed price p.

1. Derive and interpret the first-order conditions for maximizing profit.
2. Are these conditions necessary for a solution?
3. Are they sufficient to identify a global optimum?

Exercise 5.7 (Pareto optimality) Given two agents, each of which has a utility function $u_i(\mathbf{x})$ defined on some feasible set X, suppose that we wish to identify the Pareto optimal outcomes in X. Bill claims that this can be done by maximizing a weighted average of the individual utility functions, that is solving

$$\max_{\mathbf{x} \in X} \alpha u_1(\mathbf{x}) + (1 - \alpha) u_2(\mathbf{x})$$

for some $\alpha \in (0, 1)$. Under what conditions (if any) is he correct? Justify your answer.

Exercise 5.8 (Lagrangean saddle point) Suppose that f is concave and g_j are convex and there exists an $\hat{\mathbf{x}} \in X$ for which $\mathbf{g}(\hat{\mathbf{x}}) < \mathbf{0}$. Show that $\mathbf{x}^*$ is an optimal solution of

$$\max_{x \in X} f(\mathbf{x})$$
$$\text{subject to } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}$$

if and only if there exist multipliers $\boldsymbol{\lambda}^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*) \geq \mathbf{0}$ such that $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ is a saddle point of the Lagrangean $L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum \lambda_j g_j(\mathbf{x})$, that is

$$L(\mathbf{x}, \boldsymbol{\lambda}^*) \leq L(\mathbf{x}^*, \boldsymbol{\lambda}^*) \leq L(\mathbf{x}^*, \boldsymbol{\lambda}) \text{ for every } \mathbf{x} \in X \text{ and } \boldsymbol{\lambda} \geq \mathbf{0}$$

[Hint: The second inequality implies that $\mathbf{x}^*$ is feasible and also that $\boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}^*) = 0$.]