Chapter 7

Dynamic optimization

Chapter 5 deals essentially with static optimization, that is optimal choice at a single point of time. Many economic models involve optimization over time. While the same principles of optimization apply to dynamic models, new considerations arise. On the one hand, the repetitive nature of dynamic models adds additional structure to the model which can be exploited in analyzing the solution. On the other hand, many dynamic models have no finite time horizon or are couched in continuous time, so that the underlying space is infinite-dimensional. This requires a more sophisticated theory and additional solution techniques. Dynamic models are increasingly employed in economic theory and practice, and the student of economics needs to be familiar with their analysis. This need not be seen as an unrewarding chore - the additional complexity of dynamic models adds to their interest, and many interesting examples can be given.

Another factor complicating the study of dynamic optimization is the existence of three distinct approaches, all of which are used in practice. The classic approach is based on the calculus of variations, a centuries-old extension of calculus to infinite-dimensional space. This was generalized under the stimulus of the space race in the late 1950s to develop optimal control theory, the most common technique for dealing with models in continuous time. The second principle approach, dynamic programming, was developed at the same time, primarily to deal with optimization in discrete time. Dynamic programming has already been explored in some detail to illustrate the material of Chapter 2 (Example 2.32). The third approach to dynamic optimization extends the Lagrangean technique of static optimization to dynamic problems. Consequently, we call this the Lagrangean approach.

A rigorous treatment of dynamic optimization (especially optimal control theory) is quite difficult. Although many of the necessary prerequisites are contained in earlier chapters, some essential elements (such as integration) are missing. The goals of this supplementary chapter are therefore more modest. We aim to take advantage of the foundation already developed, utilizing as much as possible the optimization theory of Chapter 5. To this end, we start with the Lagrangean approach in Section 2, deriving the maximum principle for discrete time problems. We then (Section 3) extend this by analogy to continuous time problems, stating the continous-time maximum principle and illustrating its use with several examples. Section 4 takes up the dynamic programming approach.

7.1 Introduction

The basic intuition of dynamic optimization can be illustrated with a simple example of intertemporal allocation. Suppose you embark on a two-day hiking trip with w units of food. Your problem is to decide how much food to consume on the first day, and how much to save for the second day. It is conventional to label the first period 0. Therefore, let c_0 denote consumption on the first day and c_1 denote consumption on the second day. The optimization problem is

$$\max_{c_0, c_1} U(c_0, c_1)$$
$$\text{subject to } c_0 + c_1 = w$$

Clearly, optimality requires that daily consumption be arranged so as to equalize the marginal utility of consumption on the two days, that is

$$D_{c_0} U(c_0, c_1) = D_{c_1} U(c_0, c_1)$$

Otherwise, the intertemporal allocation of food could be rearranged so as to increase total utility. Put differently, optimality requires that consumption in each period be such that marginal benefit equals marginal cost, where the marginal cost of consumption in period 0 is the consumption foregone in period 1. This is the fundamental intuition of dynamic optimization - optimality requires that resources be allocated over time in such a way that there are no favorable opportunities for intertemporal trade.

Typically, the utility function is assumed to be

Separable $U(c_0, c_1) = u_0(c_0) + u_1(c_1)$ and

Stationary $U(c_0, c_1) = u(c_0) + \beta u(c_1)$

where β represents the discount rate of future consumption (Example 1.109). Then the optimality condition is

$$u'(c_0) = \beta u'(c_1)$$

Assuming that u is concave, we can deduce that

$$c_0 > c_1 \iff \beta < 1$$

Consumption is higher in the first period if future consumption is discounted.

It is straightforward to extend this model in various ways. For example, if it is possible to borrow and lend at interest rate r, the two-period optimization problem is

$$\max_{c_0, c_1} u(c_0) + \beta u(c_1)$$
$$\text{subject to } c_1 = (1 + r)(w - c_0)$$

assuming separability and stationarity. Forming the Lagrangean

$$L = u(c_0) + \beta u(c_1) + \lambda\big(c_1 - (1 + r)(w - c_0)\big)$$

the first-order conditions for optimality are

$$D_{c_0} L = u'(c_0) - \lambda(1 + r) = 0$$
$$D_{c_1} L = \beta u'(c_1) - \lambda = 0$$

Eliminating λ, we conclude that optimality requires that

$$u'(c_0) = \beta u'(c_1)(1 + r) \tag{7.1}$$

The left-hand side is the marginal benefit of consumption today. For an optimal allocation, this must be equal to the marginal cost of consumption today, which is the interest foregone $(1 + r)$ times the marginal benefit of consumption tomorrow, discounted at the rate β. Alternatively, the optimality condition can be expressed as

$$\frac{u'(c_0)}{\beta u'(c_1)} = 1 + r \tag{7.2}$$

The quantity on the left hand side is the intertemporal marginal rate of substitution. The quantity on the right can be thought of as the marginal rate of transformation, the rate at which savings in the first period can be transformed into consumption in the second period. Assuming u is concave, we can deduce from (7.2) that

$$c_0 > c_1 \iff (1 + r)\beta < 1$$

The balance of consumption between the two periods depends upon the interaction of the rate of time preference (β) and the interest rate.

Exercise 7.1 Assuming log utility $u(c) = \log c$, show that the optimal allocation of consumption is

$$c_0 = \frac{w}{1 + \beta}, \qquad c_1 = (1 + r)\frac{\beta w}{1 + \beta}$$

Note that optimal consumption in period 0 is independent of the interest rate $(D_r c_0 = 0)$.

Exercise 7.2 Suppose $u(c) = \sqrt{c}$. Show that $D_r c_0 < 0$.

To extend the model to T periods, let c_t denote consumption in period t and w_t the remaining wealth at the beginning of period t. Then

$$\begin{aligned} w_1 &= (1 + r)(w_0 - c_0) \\ w_2 &= (1 + r)(w_1 - c_1) \end{aligned}$$

and so on down to

$$w_T = (1 + r)(w_{T-1} - c_{T-1})$$

where w_0 denotes the initial wealth. The optimal pattern of consumption through times solves the problem

$$\max_{c_t, w_{t+1}} \sum_{t=0}^{T-1} \beta^t u(c_t)$$
$$\text{subject to } w_t = (1 + r)(w_{t-1} - c_{t-1}), \quad t = 1, 2, \dots, T$$

which is a standard equality constrained optimization problem. Assigning multipliers $(\lambda_1, \lambda_2, \dots, \lambda_T)$ to the T constraints, the Lagrangean is

$$L = \sum_{t=0}^{T-1} \beta^t u(c_t) - \sum_{t=1}^{T} \lambda_t \Big(w_t - (1 + r)(w_{t-1} - c_{t-1}) \Big)$$

which can be rewritten as

$$\begin{aligned} L &= \sum_{t=0}^{T-1} \beta^t u(c_t) - \sum_{t=1}^{T} \lambda_t w_t + \sum_{t=1}^{T} \lambda_t (1+r)(w_{t-1} - c_{t-1}) \\ &= \sum_{t=0}^{T-1} \beta^t u(c_t) - \sum_{t=1}^{T} \lambda_t w_t + \sum_{t=0}^{T-1} \lambda_{t+1} (1+r)(w_t - c_t) \\ &= u(c_0) - \lambda_1 (1+r)(w_0 - c_0) \\ &\quad + \sum_{t=1}^{T-1} \beta^t u(c_t) - \sum_{t=1}^{T-1} \lambda_t w_t + \sum_{t=1}^{T-1} \lambda_{t+1} (1+r)(w_t - c_t) \\ &\qquad + \lambda_T w_T \\ &= u(c_0) - \lambda_1 (1+r)(w_0 - c_0) \\ &\quad + \sum_{t=1}^{T-1} \Big(\beta^t u(c_t) - \lambda_t w_t + \lambda_{t+1} (1+r)(w_t - c_t) \Big) \\ &\qquad + \lambda_T w_T \\ &= u(c_0) - \lambda_1 (1+r)(w_0 - c_0) \\ &\quad + \sum_{t=1}^{T-1} \Big(\beta^t u(c_t) - \lambda_{t+1} (1+r) c_t + \big(\lambda_{t+1}(1+r) - \lambda_t \big) w_t \Big) \\ &\qquad + \lambda_T w_T \end{aligned}$$

The first-order necessary conditions for optimality are (Corollary 5.2.2)

$$\begin{aligned} D_{c_0} L &= u'(c_0) - \lambda_1 (1+r) = 0 \\ D_{c_t} L &= \beta^t u'(c_t) - \lambda_{t+1}(1+r) = 0, \qquad t = 1, 2, \ldots, T-1 \\ D_{w_t} L &= (1+r)\lambda_{t+1} - \lambda_t = 0, \qquad t = 1, 2, \ldots, T-1 \\ D_{w_T} L &= \lambda_T = 0 \end{aligned}$$

Together, these equations imply

$$\beta^t u'(c_t) = \lambda_{t+1}(1+r) = \lambda_t \tag{7.3}$$

in every period $t = 0, 1, \ldots, T-1$ and therefore

$$\beta^{t+1} u'(c_{t+1}) = \lambda_{t+1} \tag{7.4}$$

Substituting (7.4) in (7.3) , we get

$$\beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1})(1+r)$$

or

$$u'(c_t) = \beta u'(c_{t+1})(1+r), \; t = 0, 1, \ldots, T-1 \tag{7.5}$$

which is identical to (7.1), the optimality condition for the two-period problem. An optimal consumption plan requires that consumption be allocated through time so that marginal benefit of consumption in period t ($u'(c_t)$) is equal to its marginal cost, which is the interest foregone $(1+r)$ times the marginal benefit of consumption tomorrow discounted at the rate β. Again, we observe that whether consumption increases or decreases through time depends upon the interaction of the rate of time preference β and the interest r. Assuming u is concave, (7.5) implies that

$$c_t > c_{t+1} \iff (1+r)\beta < 1$$

The choice of the level of consumption in each period c_t has two effects. First, it provides contemporaneous utility in period t. In addition, it determines the level of wealth remaining, w_{t+1}, to provide for consumption in future periods. The Lagrange multiplier λ_t associated with the constraint

$$w_t = (1+r)(w_{t-1} - c_{t-1})$$

measures the shadow price or value of this wealth w_t at the beginning of period t. (7.3) implies that this shadow price is equal to $\beta^t u'(c_t)$. Additional wealth in period t can either be consumed or saved, and its value in these two uses must be equal. Consequently, its value must be equal to the discounted marginal utility of consumption in period t. Not that the final first-order condition is $\lambda_T = 0$. Any wealth left over is assumed to be worthless.

Exercise 7.3 An alternative approach to the multi-period optimal savings problem utilizes a single intertemporal budget constraint

$$(1+r)^T c_0 + \cdots + (1+r)^2 c_{T-2} + (1+r)c_{T-1} = (1+r)^T w_0 \tag{7.6}$$

Derive (7.6) and solve the problem of maximizing discounted total utility subject

$$\max_{c_t} \sum_{t=0}^{T-1} \beta^t u(c_t)$$

subject to this constraint.

The general finite horizon dynamic optimization problem can be depicted as

$$\begin{array}{ccccccccc} \mathbf{s}_0 & \xrightarrow{a_0} & \mathbf{s}_1 & \xrightarrow{a_1} & \mathbf{s}_2 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{T-1}} & \mathbf{s}_t \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow \\ & f_0(\mathbf{a}_0, \mathbf{s}_0) & & f_1(\mathbf{a}_1, \mathbf{s}_1) & & f_2(\mathbf{a}_2, \mathbf{s}_2) & & f_{T-1}(\mathbf{a}_{T-1}, \mathbf{s}_{T-1}) & v(\mathbf{s}_t) \end{array}$$

Starting from an initial state $\mathbf{s}_0$, the decision maker chooses some action $\mathbf{a}_0 \in A_0$ in the first period. This generates a contemporaneous return or benefit $f_0(\mathbf{a}_0, \mathbf{s}_0)$ and leads to a new state $\mathbf{s}_1$, the transition to which is determined by some function $\mathbf{g}$

$$\mathbf{s}_1 = \mathbf{g}_0(\mathbf{a}_0, \mathbf{s}_0)$$

In the second period, the decision maker choses another action $\mathbf{a}_1 \in A_1$, generating a contemporaneous return $f(\mathbf{a}_1, \mathbf{s}_1)$ and leading to a new state $\mathbf{s}_2$ to begin the third period, and so on for T periods. In each period t, the transition to the new state is determined by the *transition equation*

$$\mathbf{s}_{t+1} = \mathbf{g}_t(\mathbf{a}_t, \mathbf{s}_t)$$

The resulting sequence of choices $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{T-1}$ and implied transitions leaves a terminal state $\mathbf{s}_T$, the value of which is $v(\mathbf{s}_T)$.

Assuming separability, the objective of the decision maker is to choose that sequence of actions $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{T-1}$ which maximizes the discounted sum of the contemporaneous returns $f_t(\mathbf{a}_t, \mathbf{s}_t)$ plus the value of the terminal state $v(\mathbf{s}_T)$. Therefore, the general dynamic optimization problem is

$$\max_{\mathbf{a}_t \in A_t} \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, \mathbf{s}_t) + \beta^T v(\mathbf{s}_T) \tag{7.7}$$

$$\text{subject to } s_{t+1} = \mathbf{g}_t(\mathbf{a}_t, \mathbf{s}_t), \quad t = 0, \ldots, T-1$$

given the initial state $\mathbf{s}_0$.

The variables in this optimization problem are of two types. The action $\mathbf{a}_t$ is known as the *control variable*, since it is immediately under the control of the decision-maker, and any value $\mathbf{a}_t \in A_t$ may be chosen. In contrast, $\mathbf{s}_t$, known as the *state variable*, is determined only indirectly through the transition equation. In general, additional constraints may be imposed on the state variable. In particular, in economic models, negative values may be infeasible. As the bold face indicates, both the control and state variables can be vectors. However, to simplify the notation, we shall assume a single state variable ($s_t \in \Re$) in the rest of the chapter.

7.2 The Lagrangean approach

Stripped of its special interpretation, (7.7) is a constrained optimization problem of the type analyzed in Chapter 5, which can be solved using the Lagrangean method. In forming the Lagrangean, it is useful to multiply each constraint (transition equation) by β^{t+1}, giving the equivalent problem

$$\max_{\mathbf{a}_t \in A_t} \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, \mathbf{s}_t) + \beta^T v(\mathbf{s}_T)$$
$$\text{subject to } \beta^{t+1}\big(s_{t+1} - \mathbf{g}_t(\mathbf{a}_t, \mathbf{s}_t)\big) = 0, \quad t = 0, \dots, T-1$$

Assigning multipliers $\lambda_1, \lambda_2, \dots, \lambda_T$ to the T constraints (transition equations), the Lagrangean is

$$L = \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, s_t) + \beta^T v(s_T) - \sum_{t=0}^{T-1} \beta^{t+1} \lambda_{t+1} \big(s_{t+1} - g_t(\mathbf{a}_t, s_t)\big)$$

To facilitate derivation of the first-order conditions, it is convenient to rewrite the Lagrangean, first rearranging terms

$$\begin{aligned} L &= \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, s_t) + \sum_{t=0}^{T-1} \beta^{t+1} \lambda_{t+1} g_t(\mathbf{a}_t, s_t) - \sum_{t=0}^{T-1} \beta^{t+1} \lambda_{t+1} s_{t+1} + \beta^T v(s_T) \\ &= \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, s_t) + \sum_{t=0}^{T-1} \beta^{t+1} \lambda_{t+1} g_t(\mathbf{a}_t, s_t) - \sum_{t=1}^{T} \beta^t \lambda_t s_t + \beta^T v(s_T) \end{aligned}$$

and then separating out the first and last periods

$$\begin{aligned} L = {} & f_0(\mathbf{a}_0, s_0) + \beta \lambda_1 g_0(\mathbf{a}_0, s_0) \\ & + \sum_{t=1}^{T-1} \beta^t \Big(f_t(\mathbf{a}_t, s_t) + \beta \lambda_{t+1} g_t(\mathbf{a}_t, s_t) - \lambda_t s_t \Big) \\ & - \beta^T \lambda_T s_T + \beta^T v(s_T) \end{aligned} \tag{7.8}$$

7.2.1 Basic necessary and sufficient conditions

First, we assume that the set of feasible controls A_t is open. The gradients of the constraints are linearly independent (since each period's $\mathbf{a}_t$ appears in only one transition equation). Therefore, a necessary condition for optimality is stationarity of the Lagrangean (Theorem 5.2), which implies the following conditions. In each period $t = 0, 1, \dots, T-1$, $\mathbf{a}_t$ must be chosen such that

$$D_{\mathbf{a}_t} L = \beta^t \big(D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta \lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) \big) = 0$$

Similarly, in periods $t = 1, 2, \ldots, T-1$, the resulting s_t must satisfy

$$D_{s_t} L = \beta^t \big(D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) - \lambda_t \big) = 0$$

while the terminal state s_T must satisfy

$$D_{s_T} L = \beta^T \big(-\lambda_T + v'(s_T) \big) = 0$$

The sequence of actions $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{T-1}$ and states $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_T$ must also satisfy the transition equations

$$s_{t+1} = g_t(\mathbf{a}_t, s_t), \quad t = 0, \ldots, T-1$$

These necessary conditions can be rewritten as

$$\begin{aligned} D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) &= 0, & t &= 0, 1, \ldots, T-1 \\ D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) &= \lambda_t, & t &= 1, 2, \ldots, T-1 \quad (7.9) \\ s_{t+1} &= g_t(\mathbf{a}_t, s_t), & t &= 0, 1, \ldots, T-1 \\ \lambda_T &= v'(s_T) & & \quad (7.10) \end{aligned}$$

Stationarity of the Lagrangean is also sufficient to characterize a global optimum if the Lagrangean is concave in $\mathbf{a}_t$ and s_t (Exercise 5.20). If v is increasing, (7.10) implies that $\lambda_T \geq 0$. If in addition, f_t and g_t are increasing in s_t, (7.9) ensure that $\lambda_t \geq 0$ for every t, in which case the Lagrangean will be concave provided that f_t, g_t and v are all concave (Exercise 3.131). We summarize this result in the following theorem.

Theorem 7.1 (Finite horizon dynamic optimization) *In the finite horizon dynamic optimization problem*

$$\max_{\mathbf{a}_t \in A_t} \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, s_t) + \beta^T v(s_T)$$
$$\text{subject to } s_{t+1} = g_t(\mathbf{a}_t, s_t), \quad t = 0, \ldots, T-1$$

given the initial state s_0, suppose that

- *A_t open for every t*
- *f_t, g_t are concave and increasing in s_t*
- *v is concave and increasing.*

Then $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_T$ is an optimal solution if and only if there exist unique multipliers $(\lambda_1, \lambda_2, \ldots, \lambda_T)$ such that

$$D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) = 0, \quad t = 0, 1, \ldots, T-1 \quad (7.11)$$
$$D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) = \lambda_t, \quad t = 1, 2, \ldots, T-1 \quad (7.12)$$
$$s_{t+1} = g_t(\mathbf{a}_t, s_t), \quad t = 0, 1, \ldots, T-1 \quad (7.13)$$
$$\lambda_T = v'(s_T) \quad (7.14)$$

To interpret these conditions, observe that a marginal change in $\mathbf{a}_t$ in period t has two effects. It changes the instantaneous return in period t by $D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t)$. In addition, it has future consequences, changing the state in the next period s_{t+1} by $D\mathbf{a}_t g_t(\mathbf{a}_t, s_t)$, the value of which is measured by the Lagrange multiplier λ_{t+1}. Discounting to the current period, $D_{\mathbf{a}_t} f_(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{\mathbf{a}_t} g_(\mathbf{a}_t, s_t)$ measures the total impact of a marginal change in $\mathbf{a}_t$. The first necessary condition (7.11) requires that, in every period, $\mathbf{a}_t$

must be chosen optimally, taking account of both present and future consequences of a marginal change in $\mathbf{a}_t$.

The second necessary condition (7.12) governs the evolution of λ_t, the shadow price of s_t. A marginal change s_t has two effects. It changes the instantaneous return in period t by $D_{s_t} f_t(\mathbf{a}_t, s_t)$. In addition, it alters the attainable state in the next period s_{t+1} by $D_{s_t} g_t(\mathbf{a}_t, s_t)$, the value of which is $\lambda_{t+1} D_{a_t} g_t(\mathbf{a}_t, s_t)$. Discounting to the current period, the total impact of marginal change in s_t is given by $D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t)$, which is precisely what is meant by the shadow price λ_t of s_t. The second necessary condition (7.12) requires that the shadow price in each period λ_t correctly measures the present and future consequences of a marginal change in s_t. (7.13) is just the transition equation, while the final condition (7.14) states that the resulting shadow price of $\mathbf{s}_T$ is must be equal to its marginal value $v'(s_T)$.

The necessary and sufficient conditions (7.11) to (7.14) constitute a simultaneous system of $3T$ equations in $3T$ unknowns, which in principle can be solved for the optimal solution (Example 7.1). In general, though, the split boundary conditions (s_0 given, $\lambda_T = v'(s_T)$) precludes a simple recursive solution. This does not usually pose a problem in economic models, where we are typically interested in characterising the optimal solution rather than solving particular problems (Examples 7.2 and 7.3).

Example 7.1 (Closing the mine) Suppose you own a mine. Your mining licence will expire in three years and will not be renewed. There are known to be 128 tons of ore remaining in the mine. The price is fixed at \$1 a ton. The cost of extraction is q_t^2/x_t where q_t is the rate of extraction and x_t is the stock of ore. Ignoring discounting for simplicity ($\beta = 1$), the optimal production plan solves

$$\max_{q_t, x_t} \sum_{t=0}^{3} \left(1 - \frac{q_t}{x_t}\right) q_t$$
$$\text{subject to } x_{t+1} = x_t - q_t, \quad t = 0, 1, 2$$

The Lagrangean is

$$L = \sum_{t=0}^{3} \left(1 - \frac{q_t}{x_t}\right) q_t - \lambda_{t+1}(x_{t+1} - x_t + q_t)$$

The first-order conditions are

$$\begin{aligned} D_{q_t} L &= 1 - 2\frac{q_t}{x_t} - \lambda_{t+1} = 0, \quad t = 0, 1, 2 \\ D_{x_t} L &= \left(\frac{q_t}{x_t}\right)^2 - \lambda_t + \lambda_{t+1} = 0, \quad t = 1, 2 \\ x_{t+1} &= x_t - q_t, \quad t = 0, 1, 2 \\ \lambda_3 &= 0 \end{aligned}$$

Let $z_t = 2q_t/x_t$ (marginal cost). Substituting, the first-order conditions become

$$\begin{aligned} z_t + \lambda_{t+1} &= 1, \quad t = 0, 1, 2 \\ \lambda_t &= \lambda_{t+1} + \frac{1}{4} z_t^2, \quad t = 1, 2 \\ \lambda_3 &= 0 \end{aligned}$$

The left-hand side of the first equation is the extra cost of selling an additional unit in period t, comprising the marginal cost of extraction z_t plus the opportunity cost of having one less unit to sell in the subsequent period, which is measured by the shadow price of the stock in period $t+1$. Optimality requires that this cost be equal to the price of selling an additional unit, which is 1.

The first-order conditions provide a system of difference equations, which in this case can be solved recursively. The optimal plan is

t	x_t	q_t	z_t	λ_{t+1}
0	128	39	0.609375	0.390625
1	89	33.375	0.75	0.25
2	55.625	27.8125	1	0

Example 7.2 (Optimal economic growth) A finite horizon verion of the optimal economic growth model (Example 2.33)

$$\max_{c_t} \sum_{t=0}^{T-1} \beta^t u(c_t) + \beta^T v(k_T)$$
$$\text{subject to } k_{t+1} = F(k_t) - c_t$$

where c is consumption, k is capital, $F(k_t)$ is the total supply of goods available at the end of period t, comprising current output plus undepreciated capital, and $v(k_T)$ is the value of the remaining capital at the end of the planning horizon. Setting $\mathbf{a}_t = c_t$, $s_t = k_t$, $f(a_t, s_t) = u(c_t)$, $g(a_t, s_t) = F(k_t) - c_t$. It is economically reasonable to assume that u is concave, and that F and v are concave and increasing, in which case the optimality conditions (7.11) to (7.14) are both necessary and sufficient. That is, an optimal plan satisfies the equations

$$u'(c_t) = \beta\lambda_{t+1} \qquad t = 0, 1, \ldots, T-1 \tag{7.15}$$
$$\lambda_t = \beta\lambda_{t+1}F'(k_t) \qquad t = 1, 2, \ldots, T-1 \tag{7.16}$$
$$k_{t+1} = F(k_t) - c_t \qquad t = 0, 1, \ldots, T-1 \tag{7.17}$$
$$\lambda_T = v'(k_T) \tag{7.18}$$

To interpret these conditions, observe that, in any period, output can be either consumed or saved, in accordance with the transition equation (7.17). The marginal benefit of additional consumption in period t is $u'(c_t)$. The future consequence of additional consumption is a reduction in capital available for the subsequent period, the value of which, discounted to period t, is $\beta\lambda_{t+1}$. This is the marginal cost of additional consumption in period t. The first necessary condition (7.15) for an optimal plan requires that consumption in each period be chosen so that the marginal benefit of additional consumption is equal to its marginal cost.

Now focus on period $t+1$, when (7.15) and (7.16) require

$$u'(c_{t+1}) = \beta\lambda_{t+2} \text{ and } \lambda_{t+1} = \beta\lambda_{t+2}F'(k_{t+1}) \tag{7.19}$$

The impact of additional capital in period $t+1$ is increased production $F'(k_{t+1})$. This additional production could be saved for the subsequent period, in which case it would be worth $\beta\lambda_{t+2}F'(k_{t+1})$. Alternatively, the additional production could be consumed, in which case it would be worth $u'(c_{t+1})F'(k_{t+1})$. Together, equations (7.19) imply that

$$\beta\lambda_{t+1} = \beta u'(c_{t+1})F'(k_{t+1})$$

But (7.15) implies that this is equal to the the marginal benefit of consumption in period t, that is

$$u'(c_t) = \beta u'(c_{t+1})F'(k_{t+1}) \tag{7.20}$$

which is known as the *Euler equation*. The left-hand side is the marginal benefit of consumption in period t, while the right-hand side is the marginal cost, where the marginal cost is measured by marginal utility of potential consumption foregone ($u'(c_{t+1})F'(k_{t+1})$ discounted one period.

The Euler equation (7.20) can be rearranged to give

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = F'(k_{t+1})$$

The left-hand side of this equation is the intertemporal marginal rate of substitution in consumption, while the right-hand side is the marginal rate of transformation in production, the rate of which additional capital can be transformed into additional output.

Subtracting $u'(c_{t+1})$ from both sides, the Euler equation (7.20) can be expressed as

$$u'(c_t) - u'(c_{t+1}) = (\beta F'(k_{t+1}) - 1)u'(c_{t+1})$$

Assuming that c is concave, this implies

$$c_{t+1} \gtreqless c_t \iff \beta F'(k_{t+1}) \gtreqless 1$$

Whether consumption is increasing or decreasing under the optimal plan depends on the balance between technology and the rate of time preference.

The Euler equation (7.20) determines relative consumption between successive periods. The actual level of the optimal consumption path $c_0, c_1, \ldots, c_{T-1}$ is determined by the initial capital k_0 and by the requirement (7.18) that the shadow price of capital in the final period λ_T be equal to the marginal value of the terminal stock $v'(k_T)$

Exercise 7.4 (Optimal savings) Derive (7.5) by applying Theorem 7.1 to the optimal savings problem

$$\max \sum_{t=0}^{T-1} \beta^t u(c_t)$$
$$\text{subject to } w_{t+1} = (1+r)(w_t - c_t)$$

analysed in the previous section.

Remark 7.1 (Form of the Lagrangean) In forming the Lagrangean (7.8), we first multiplied each transition equation by β^{t+1}. If we do not do this, the Lagrangean is

$$\begin{aligned} L = {} & f_0(\mathbf{a}_0, s_0) + \mu_1 g_0(\mathbf{a}_0, s_0) \\ & + \sum_{t=1}^{T-1} \left(\beta^t f_t(\mathbf{a}_t, s_t) + \mu_{t+1} g_t(\mathbf{a}_t, s_t) - \mu_t s_t\right) \\ & - \mu_T s_T + \beta^T v(s_T) \end{aligned}$$

and the necessary conditions for optimality become

$$\beta^t D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \mu_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) = 0, \qquad t = 0, 1, \ldots, T-1 \tag{7.21}$$
$$\beta^t D_{s_t} f_t(\mathbf{a}_t, s_t) + \mu_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) = \mu_t, \qquad t = 1, 2, \ldots, T-1 \tag{7.22}$$
$$s_{t+1} = g_t(\mathbf{a}_t, s_t), \quad t = 0, 1, \ldots, T-1 \tag{7.23}$$
$$\mu_T = \beta^T v'(s_T) \tag{7.24}$$

These are precisely equivalent to (7.11) to (7.14), except for the interpretation of μ (see Remark 5.3). The Lagrange multiplier μ_t in (7.21) to (7.24) measures the value of the state variable discounted to the first period (the *initial value multiplier*). In contrast, λ_t in (7.11) to (7.14), which measures the value of s_t in period t, is called the *current value multiplier*.

Exercise 7.5 Apply the necessary conditions (7.21) to (7.24) to the optimal growth model (Example 7.2), and show that they imply the same optimal plan.

7.2.2 Transversality conditions

The terminal condition (7.14) determining the value of the Lagrange multiplier at the end of the period

$$\lambda_T = v'(S_T)$$

is known as a *transversality condition.* This specific transversality condition is the one appropriate to problems in which terminal value of the state variable (s_T) is free, as in (7.7) and examples 7.1 and 7.2. Note in particular, where $v(s_T, T) = 0$, optimality requires that $\lambda_T = 0$.

In other problems, the terminal value of the state variable (s_T) may be specified, or at least constrained to lie in a certain set. In yet other problems (for example optimal search), the terminal time T itself may be endogenous, to be determined as part of the solution. In each case, the transversality condition must be modified appropriately. The following table summarizes the transversality conditions for some common cases.

Table 7.1: Transversality conditions

Terminal condition	Transversality condition
s_T fixed	None
s_T free	$\lambda_T = V_{s_T}$
$s_T \geq \bar{s}$	$\lambda_T \geq 0$ and $\lambda_T(s_T - \bar{s}) = 0$

Exercise 7.6 Consider the finite horizon dynamic optimization problem (7.7) with a terminal constraint

$$\max_{\mathbf{a}_t \in A_t} \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, s_t)$$
$$\text{subject to } s_{t+1} = g_t(\mathbf{a}_t, s_t), \quad t = 0, \ldots, T-1$$
$$s_T \geq \bar{s}$$

given the initial state s_0. Assume for every $t = 0, \ldots, T-1$

- A_t is open
- f_t, g_t are concave and increasing in s_t

Show that $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_T$ is an optimal solution if and only if there exists unique multipliers $\lambda_1, \lambda_2, \ldots, \lambda_T$ satisfying (7.11) to (7.12) together with

$$\lambda_T \geq 0 \text{ and } \lambda_T(s_T - \bar{s}) = 0$$

7.2.3 Nonnegative variables

Both control and state variable in economic models are usually required to be nonnegative, in which case the necessary conditions should be modified as detailed in the following corollary.

Corollary 7.1.1 (Nonnegative variables) *In the finite horizon dynamic optimization problem*

$$\max_{\mathbf{a}_t \geq \mathbf{0}} \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, s_t) + \beta^T v(s_T)$$
$$\text{subject to } s_{t+1} = \mathbf{g}_t(\mathbf{a}_t, s_t) \geq 0, \quad t = 0, \dots, T-1$$

given the initial state s_0, suppose that

- f_t, g_t *are concave in* $\mathbf{a}$ *and* s *and increasing in* s
- v *is concave and increasing.*

Then $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_T$ is an optimal solution if and only if there exist unique multipliers $(\lambda_1, \lambda_2, \dots, \lambda_T)$ such that

$$\begin{aligned}
&D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) \square 0, \qquad a_t \geq 0, \\
&\qquad (D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t)) a_t = 0, \qquad t = 0, 1, \dots, T-1 \\
&D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) \square \lambda_t, \qquad s_t \geq 0 \\
&\qquad (D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) - \lambda_t) s_t = 0, \qquad t = 1, 2, \dots, T-1 \\
&s_{t+1} = g_t(\mathbf{a}_t, s_t), \qquad t = 0, 1, \dots, T-1 \\
&\lambda_T \geq v'(s_T), \qquad s_T \geq 0, \qquad \lambda_T s_T = 0
\end{aligned}$$

Exercise 7.7 Prove corollary 7.1.1.

Example 7.3 (Exhaustible resources) Consider a monopolist extracting an exhaustible resource, such as a mineral deposit. Let x_t denote the size of the resource at the beginning of period t, and let q_t quantity extracted during in period t. Then, the quantity remaining at the beginning of period $t+1$ is $x_t - q_t$. That is, the stock evolves according to the simple transition equation

$$x_{t+1} = x_t - q_t$$

If demand is determined by the known (inverse) demand function $p_t(q_t)$ and extraction incurs a constant marginal cost of c per unit, the net profit in each period is $(p_t(q_t) - c)q_t$. The monopolist's objective is to maximize total discounted profit

$$\Pi = \sum_{t=0}^{T-1} \beta^t \big(p_t(q_t) - c\big) q_t$$

Note that this objective function is separable but not stationary, since demand can vary through time. Since negative quantities are impossible, both q_t and x_t must be nonnegative. Summarizing, the monopolist's optimization problem is

$$\max_{q_t \geq 0} \sum_{t=0}^{T-1} \beta^t \big(p_t(q_t) - c\big) q_t$$
$$x_{t+1} = x_t - q_t \geq 0$$

given an initial stock x_0.

Setting $a_t = q_t$, $s_t = x_t$, $f_t(a_t, s_t) = (p_t(q_t) - c)q_t$, $g_t(a_t, s_t) = x_t - q_t$ and $v(s_T) = 0$, and letting $m_t(q_t) = p_t(q_t) + p_t'(q_t)q_t$ denote marginal revenue in period t, we observe that

$$D_{\mathbf{a}} f_t(\mathbf{a}_t, s_t) = \big(p_t(q_t) + p_t'(q_t)q_t - c\big) = \big(m_t(q_t) - c\big)$$

Applying Corollary 7.1.1, the necessary conditions for optimality are

$$\begin{array}{lll} \big(m_t(q_t) - c\big) \leq \beta\lambda_{t+1} & q_t \geq 0 & \Big(\big(m_t(q_t) - c\big) - \beta\lambda_{t+1}\Big)q_t = 0 \quad (7.25) \\ \beta\lambda_{t+1} \leq \lambda_t & x_t \geq 0 & (\beta\lambda_{t+1} - \lambda_t)x_t = 0 \quad (7.26) \\ \lambda_T \geq 0 & \mathbf{x}_T \geq 0 & \lambda_T x_T = 0 \end{array}$$

where (7.25) holds for all periods $t = 0, 1, \ldots, T-1$ and (7.26) holds for periods $t = 1, 2, \ldots, T-1$. Provided marginal revenue is decreasing, these conditions are also sufficient. In interpreting these conditions, we observe that there are two cases.

Case 1 In the first case, the initial quantity is so high that it is not worthwhile extracting all the resource in the available time leaving $x_T > 0$ which implies that $\lambda_T = 0$. Nonnegativity ($q_t \geq 0$) implies that $x_t > 0$ for all t, which by (7.26) implies that $\lambda_0 = \lambda_1 = \cdots = \lambda_T = 0$. Then (7.25) implies that $m_t(q_t) \leq c$ in every period, with $m_t(q_t) = c$ if $q_t > 0$. That is, the monopolist should produce where marginal revenue equals marginal cost, provided the marginal revenue of the first unit exceeds its marginal cost. In effect, there is no resource constraint, and the dynamic problem reduces to a sequence of standard single period monopoly problems.

Case 2 In the more interesting case, the resource is scarce and it is worthwhile extracting the entire stock ($x_T = 0$). For simplicity, assume that output is positive ($q_t > 0$) in every period. (The general case is analysed in Exercise 7.8). Then (7.25) implies

$$m_t(q_t) = c + \beta\lambda_{t+1} \text{ for every } \; t = 0, 1, \ldots, T-1$$

Again, optimality requires producing where marginal revenue equals marginal cost, but in this case the marginal cost of production in period t includes both the marginal cost of extraction c plus the opportunity cost of the reduction in stock available for subsequent period, which is measured by the shadow price of the remaining resource λ_{t+1} discounted to the current period.

In particular, in the subsequent period $t+1$, we have

$$m_{t+1}(q_{t+1}) - c = \beta\lambda_{t+2}$$

But (7.26) implies (since $x_{t+1} > 0$) that $\beta^2\lambda_{t+2} = \beta\lambda_{t+1}$ so that

$$\beta(m_{t+1}(q_{t+1}) - c) = \beta^2\lambda_{t+2} = \beta\lambda_{t+1} = m_t(q_t) - c$$

So that an optimal extraction plan is characterized by

$$m_t(q_t) - c = \beta\big(m_{t+1}(q_{t+1}) - c)\big) \tag{7.27}$$

The left-hand side is the net profit from selling an additional unit in the current period. The right-hand side is the opportunity cost of selling an additional unit in the current period, which is foregone opportunity to sell an additional unit in the subsequent period. Extraction should be organized through time so that no profitable opportunity to reallocate production between adjacent periods remains. Note that this is precisely analogous to the condition for the optimal allocation of consumption we obtained in section 1.

Exercise 7.8 Extend the analysis of Case 2 in example 7.3 to allow for the possibility that it is optimal to extract nothing ($q_t = 0$) in some periods.

Remark 7.2 (Hotelling's rule) Hotelling (1931) showed that, in a competitive industry, the price of an exhaustible resource must change so that net rents increase at the rate of interest r, that is

$$\frac{p_{t+1} - c}{p_t - c} = 1 + r \text{ for every } t$$

a result known as *Hotelling's rule*. Otherwise, there would be opportunity for profitable arbitrage. For a profit-maximizing monopolist (in the absence of uncertainty), the discount rate is $\beta = 1/(1+r)$ and (7.27) implies

$$\frac{m_{t+1} - c}{m_t - c} = \frac{1}{\beta} = 1 + r \text{ for every } t \tag{7.28}$$

In other words, the monopolist should arrange production so that the marginal profit rises at the rate of interest, since otherwise it would be profitable to rearrange production through time.

Exercise 7.9 (Conservation and market structure) Will a monopoly extract an exhaustible resource at a faster or slower rate than a competitive industry? Assume zero extraction cost.

[Hint: Compare the rate of price change implied by (7.28) with Hotelling's rule. Note that marginal revenue can be rewritten as

$$m_t(q_t) = p_t(q_t) + p'(q_t)q_t = p_t(1 + p'(q_t)\frac{q_t}{p_t}) = p_t(1 + \frac{1}{\epsilon_t})$$

where ϵ_t is the elasticity of demand in period t.]

7.2.4 The Maximum principle

Some economy of notation can be made by defining the *Hamiltonian* by

$$H_t(\mathbf{a}_t, s_t, \lambda_{t+1}) = f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1}g_t(\mathbf{a}_t, s_t) \tag{7.29}$$

Then the Lagrangean (7.8) becomes

$$L = H_0(\mathbf{a}_0, s_0, \lambda_1) + \sum_{t=1}^{T-1} \beta^t \big(H_t(\mathbf{a}_t, s_t, \lambda_{t+1}) - \lambda_t s_t\big) - \beta^T \lambda_T s_T + \beta^T v(s_T)$$

Assuming A_t is open, stationarity requires

$$\begin{aligned}
D_{\mathbf{a}_t} L &= \beta^t D_{\mathbf{a}_t} H_t(\mathbf{a}_t, s_t, \lambda_{t+1}) = 0, \quad t = 0, 1, \dots, T-1 \\
D_{s_t} L &= \beta^t \big(D_s H_t(a_t, s_t, \lambda_{t+1}) - \lambda_t\big) = 0, \quad t = 1, 2, \dots, T-1 \\
D_{S_T} L &= \beta^T \big(-\lambda_T + V'(S_T)\big) = 0
\end{aligned}$$

These necessary conditions can be rewritten as

$$\begin{aligned}
D_{\mathbf{a}_t} H_t(\mathbf{a}_t, s_t, \lambda_{t+1}) &= 0, \quad t = 0, 1, \dots, T-1 \\
D_s H_t(a_t, s_t, \lambda_{t+1}) &= \lambda_t, \quad t = 1, 2, \dots, T-1 \\
\lambda_T &= v'(S_T)\big) = 0
\end{aligned}$$

Of course, the optimal plan must also satisfy the transition equation

$$s_t = g_t(\mathbf{a}_t, s_t), \quad t = 0, 1, \dots, T-1$$

Under the same assumptions as theorem 7.1, stationarity is also sufficient for a global optimum (Exercise 5.20).

It is not merely that the Hamiltonian enables economy of notation. Its principal merit lies in its economic interpretation. The Hamiltonian

$$H_t(\mathbf{a}_t, s_t, \lambda_{t+1}) = f(a_t, s_t) + \beta\lambda_{t+1}g(a_t, s_t)$$

measures the total return in period t. The choice of $\mathbf{a}_t$ in period t affects the total return in two ways. The first term $f(\mathbf{a}_t, s_t)$ reflects the direct effect of choosing a_t in period t. The second term $\lambda_{t+1}g(a_t, s_t)$ measures change in the value of state variable, the ability to provide returns in the future. The Hamiltonian augments the single period return $f(a_t, s_t)$ to account for the future consequences of current decisions, aggregating the direct and indirect effects of the choice of $\mathbf{a}_t$ in period t.

The first-order condition

$$D_{\mathbf{a}_t} H_t(\mathbf{a}_t, s_t, \lambda_{t+1}) = 0, \quad t = 0, 1, \ldots, T-1$$

characterizes an interior maximum of the Hamiltonian along the optimal path. The principal applies more generally. For example, if the actions are constrained to some set A_t, the previous equation should be replaced by

$$\max_{\mathbf{a}_t \in A} H_t(\mathbf{a}_t, s_t, \lambda_{t+1}), \quad t = 0, 1, \ldots, T-1$$

The *Maximum Principal* prescribes that, along the optimal path, a_t should be chosen in such a way as to maximize the total benefits in each period. In a limited sense, the Maximum principle transforms a dynamic optimization problem into a sequence of static optimization problems. These static problems are related by two intertemporal equations - the transition equation and the corresponding equation determining the evolution of the shadow price λ_t.

Corollary 7.1.2 (Maximum principle) *In the finite horizon dynamic optimization problem*

$$\max_{\mathbf{a}_t \in A_t} \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, s_t) + \beta^T v(s_T)$$
$$\textit{subject to } s_{t+1} = g_t(\mathbf{a}_t, s_t), \quad t = 0, \ldots, T-1$$

given the initial state s_0, suppose that

- *f_t, g_t are concave and increasing in s_t*
- *v is concave and increasing.*

Then $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_T$ is an optimal solution if and only if

$$\max_{\mathbf{a}_t \in A} H_t(\mathbf{a}_t, s_t, \lambda_{t+1}), \quad t = 0, 1, \ldots, T-1 \tag{7.30}$$
$$D_s H_t(a_t, s_t, \lambda_{t+1}) = \lambda_t, \quad t = 1, 2, \ldots, T-1$$
$$s_t = g_t(\mathbf{a}_t, s_t), \quad t = 0, 1, \ldots, T-1 \tag{7.31}$$
$$\lambda_T = v'(S_T))$$

where $H_t(\mathbf{a}_t, s_t, \lambda_{t+1}) = f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1}g_t(\mathbf{a}_t, s_t)$ is the Hamiltonian.

Proof. The proof for A_t open is given above. The general case is given in Cannon, Cullum and Polak (1970). □

Remark 7.3 Observing that

$$g_t(\mathbf{a}_t, s_t) = D_{\beta\lambda_{t+1}} H(a_t, s_t, \lambda_{t+1})$$

the transition equation can be written as

$$s_{t+1} = D_{\beta\lambda_{t+1}} H(a_t, s_t, \lambda_{t+1})$$

Suppressing function arguments for clarity, the necessary and sufficient conditions can be written very compactly as

$$\begin{aligned}
&\max_{\mathbf{a}} H, && t = 0, 1, \ldots, T-1 \\
s_{t+1} &= D_{\beta\lambda_{t+1}} H, && t = 0, 1, \ldots, T-1 \\
\lambda_t &= D_s H, && t = 1, 2, \ldots, T-1 \\
\lambda_T &= v'
\end{aligned}$$

Example 7.4 (Optimal economic growth) In the optimal growth problem (Example 7.2, the Hamiltonian is

$$H(c_t, k_t, \lambda_{t+1}) = u(c_t) + \beta\lambda_{t+1}(F(k_t) - c_t)$$

which yields immediately the optimality conditions

$$\begin{aligned}
D_c H &= u'(c_t) - \beta\lambda_{t+1} = 0, \quad t = 0, 1, \ldots, T-1 \\
k_{t+1} &= D_{\beta\lambda_{t+1}} H = F(k_t) - c_t, \quad t = 0, 1, \ldots, T-1 \\
\lambda_t &= D_k H = \beta\lambda_{t+1} F'(k_t), \quad t = 1, 2, \ldots, T-1 \\
\lambda_T &= v'(k_T)
\end{aligned}$$

Remark 7.4 (Form of the Hamiltonian) The Hamiltonian defined in equation (7.29) is known as the *current value Hamiltonian* since it measures the total return in period t. Some authors use the *initial value Hamiltonian*

$$H_t(\mathbf{a}_t, s_t, \mu_{t+1}) = \beta^t f_t(\mathbf{a}_t, s_t) + \mu_{t+1} g_t(\mathbf{a}_t, s_t) \tag{7.32}$$

where μ_{t+1} is the present value multiplier (Remark 7.1). This measures the total return in period t discounted to the initial period, and yields the equivalent optimality conditions

$$\begin{aligned}
&\max_{\mathbf{a}} H, && t = 0, 1, \ldots, T-1 \\
s_{t+1} &= D_{\mu_{t+1}} H, && t = 0, 1, \ldots, T-1 \\
\mu_t &= D_s H, && t = 1, 2, \ldots, T-1 \\
\mu_T &= v'
\end{aligned}$$

7.2.5 Infinite horizon problems

Many problems have no fixed terminal date and are more appropriately or conveniently modeled as infinite horizon problems, so that (7.7) becomes

$$\max_{\mathbf{a}_t \in A_t} \sum_{t=0}^{\infty} \beta^t f_t(a_t, s_t) \tag{7.33}$$

$$\text{subject to } s_{t+1} = g_t(a_t, s_t), \quad t = 0, 1, \ldots$$

given s_0. To ensure that the total discounted return is finite, we assume that f_t is bounded for every t and $\beta < 1$.

An optimal solution to (7.33) must also be optimal over any finite period, provided the future consequences are correctly taken into account. That is, (7.33) is equivalent to

$$\max_{\mathbf{a}_t \in A_t} \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T v_T(s_T)$$
$$\text{subject to } s_{t+1} = g_t(a_t, s_t), \quad t = 0, 1, \ldots, T-1$$

where

$$v_T(s_T) = \max_{\mathbf{a}_t \in A_t} \sum_{t=T}^{\infty} \beta^{t-T} f_t(a_t, s_t) \text{ subject to } s_{t+1} = g_t(a_t, s_t), \quad t = T, T+1, \ldots$$

This is an instance of the principle of optimality to be discussed in Section 4. It follows that the infinite horizon problem (7.33) must satisfy the same intertemporaral optimality conditions as its finite horizon cousin, namely

$$\begin{aligned} D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) &= 0, & t &= 0, 1, \ldots \\ D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) &= \lambda_t, & t &= 1, 2, \ldots \\ s_{t+1} &= g_t(\mathbf{a}_t, s_t), & t &= 0, 1, \ldots \end{aligned}$$

Example 7.5 (Investment in the competitive firm) Consider a competitive firm producing a single output with two inputs, "capital" (k) and "labor" (l) according to the production function $f(k, l)$. In the standard static theory of the firm, the necessary conditions to maximize the firm's profit (Example 5.11)

$$\max \Pi = pf(k, l) - wl - rk$$

are

$$pf_k = r \text{ and } pf_l = w$$

where p is the price of output, r the price of capital services, w the price of labor (wage rate), and f_k and f_l are the marginal products of capital and labour respectively.

Suppose that one of the inputs, "capital" (k), is long lived. Then we need to consider multiple periods, allowing for discounting and depreciation. Assume that capital depreciates at rate δ so that

$$k_{t+1} = (1 - \delta)k_t + I_t$$

where k_t is the stock of capital and I_t the firm's investment in period t. The firm's net revenue in period t is

$$\pi_t = p_t f(k_t, l_t) - w_t l_t - q_t I_t$$

where q is the price of new capital. Assuming that the firm's objective is maximise the present value of its profits and there is no final period, its optimization problem is

$$\max \sum_{t=0}^{\infty} \beta^t \pi_t = \sum_{t=0}^{\infty} \beta^t \big(p_t f(k_t, l_t) - w_t l_t - qI_t\big)$$

subject to

$$k_0 = \bar{k}_0, \qquad k_{t+1} = (1 - \delta)k_t + I_t, \qquad t = 0, 1, 2, \ldots$$

where $\bar{k}_0$ is the initial capital and β is the discount factor. Setting $\mathbf{a}_t = I_t$, $s_t = k_t$, $f(\mathbf{a}_t, s_t) = \big(p_t f(k_t, l_t) - w_t l_t - qI_t\big)$, $g(\mathbf{a}_t, s_t) = (1-\delta)k_t + I_t$ the necessary conditions for optimal production and investment include

$$-q + \beta\lambda_{t+1} = 0 \tag{7.34}$$
$$p_t f_l(k_t, l_t) - w_t = 0 \tag{7.35}$$
$$p_t f_k(k_t, l_t) + \beta\lambda_{t+1}(1-\delta) = \lambda_t$$
$$k_{t+1} = (1-\delta)k_t + I_t$$

where f_k and f_l are the marginal products of capital and labour respectively. Equation (7.35) requires

$$p_t f_l(k_t, l_t) = w_t$$

in every period, the standard marginal productivity condition for labour. Since there are no restrictions on the employment of labour, the firm employs the optimal quantity of labour (given its capital stock) in each period.

Equation (7.34) implies

$$\beta\lambda_{t+1} = q$$

and therefore λ is constant

$$\lambda_t = \lambda_{t+1} = \frac{q}{\beta}$$

Substituting into (7.35) yields

$$p_t f_k(k_t, l_t) + (1-\delta)q = \frac{q}{\beta}$$

or

$$p_t f_k(k_t, l_t) = \frac{q}{\beta} - (1-\delta)q$$

Letting $\beta = 1/(1+r)$, we get

$$p_t f_k(k_t, l_t) = (1+r)q - (1-\delta)q = (r+\delta)q$$

The right hand side $(r+\delta)q$ is known as the *user cost* of capital, the sum of the interest cost and the depreciation. This condition requires that investment be determined so the marginal benefit of capital pf_k be equal to its user cost in each period. These necessary conditions are also sufficient provided that the production function f is concave.

Exercise 7.10 Modify the model in the previous example to allow for the possibility that investment is irreversible so that $I_t \geq 0$ in every period. Derive and interpret the necessary conditions for an optimal policy.

The infinite horizon precludes using backward induction to solve for the optimal solution. Where the problem is stationary (f, g independent of t), it may be reasonable to assume that the optimal solution converges to a *steady state* in which variables are constant, that is

$$\mathbf{a}_t = \mathbf{a}^* \quad s_t = s^* \quad \lambda_t = \lambda^* \text{ for every } \ t \geq T$$

satisfying the necessary conditions

$$D_{\mathbf{a}^*} f(\mathbf{a}^*, s^*) + \beta\lambda^* D_{\mathbf{a}^*} g(\mathbf{a}^*, s^*) = 0$$
$$D_{s^*} f(\mathbf{a}^*, s^*) + \beta\lambda^* D_{s^*} g(\mathbf{a}^*, s^*) = \lambda^*$$
$$s^* = g(\mathbf{a}^*, s^*)$$

These conditions can then be used to analyze the properties of the steady state.

Example 7.6 (Golden rule) In the optimal economic growth model (Example 2.33), a steady state requires

$$\lambda^* = \beta\lambda^* f'(k^*)$$

That is, the steady state capital stock is determined by

$$\beta f'(k^*) = 1$$

Under the *golden rule* of growth, capital stock is set a level which maximizes steady state consumption. In a steady state (c^*, k^*), consumption is

$$c^* = f(k^*) - k^*$$

which is maximized where

$$f'(k^*) = 1$$

To achieve this target level of capital requires sacrificing current consumption. The optimal growth policy discounts the future, and sets a more modest target

$$\beta f'(k^*) = 1$$

which promises a lower level of steady state consumption. In other words, an optimal policy trades off potential future consumption against higher current consumption. This is known as the *modified golden rule*.

Exercise 7.11 (The Alcoa case) In the celebrated *Alcoa Case* (1945), Judge Learned Hand ruled that Alcoa constituted an illegal monopoly since it controlled 90% of domestic aluminium production. Several economists criticised this decision, arguing that the competitive aluminium recycling industry would restrain Alcoa from abusing its dominant position. To adequately examine this issue requires an intertemporal model.

1. Assume that aluminium lasts only one period. At the end of the period, it is either recycled or scrapped. Let q_{t-1} denote the stock of aluminium available in period $t-1$ and let x_t denote the fraction which is recycled for use in period t. Then

$$q_t = y_t - x_t q_{t-1}$$

   where y_t is the production of new aluminium in period t. Let $C(x_t)$ denote the cost of recycling. Assume that C is a strictly convex, increasing function with $C(0) = 0$ and $C(1) = \infty$. Assuming that recycling is competitive, show that the fraction x of output recycled is an increasing function of the price of aluminium p, that is

$$x_t = x(p_t), \qquad x' > 0$$

2. Suppose that new aluminium is produced by a monopolist at constant marginal cost c. Assume that there is a known inverse demand function $p_t = P(q_t)$. The monoplist's objective is maximize the present discounted value of profits

$$\max \sum_{t=0}^{\infty} \beta^t (P(q_t) - c) y_t$$
$$\text{where } q_t = y_t - x_t q_{t-1}$$

where β is the discount factor. Show that in a steady state, the optimal policy satisfies

$$(p-c)(1-\beta x - x'P'q) = -(1-x)P'q$$

You can assume that the second-order conditions are satisfied. Note that x is a function of p.

[Hint: This is a case in which it may be easier to proceed from first principles rather than try and fit the model into the standard formulation.]

3. Deduce that $p > c$. That is, recycling does not eliminate the monopolist's market power. In fact $p \to c$ if and only if $x \to 1$.

4. Show however that recycling does limit the monopolist's market power and therefore increase welfare.

7.3 Continuous Time

So far, we have divided time into discrete intervals, such as days, months, or years. While this is appropriate for many economic models, it is not suitable for most physical problems. Problems involving motion in space, such as the guiding a rocket to the moon, need to be expressed in continuous time. Consequently, dynamic optimization has been most fully developed in this framework, where additional tools of analysis can be exploited. For this reason, it is often useful to adopt the continuous time framework in economic models.

Remark 7.5 (Discounting in continuous time) The discount rate β in the discrete time model (7.7) can be thought of as the present value of $1 invested at the interest rate r. That is, to produce a future return of $1 when the interest rate is r

$$\beta = \frac{1}{1+r}$$

needs to be invested, since this amount will accrue to

$$\beta(1+r) = 1$$

after one period. However, suppose interest is accumulated n times during period, with r/n earned each sub-period and the balance compounded. Then, the present value is

$$\beta = \frac{1}{(1+r/n)^n}$$

since this amount will accrue to

$$\beta\left(1+\frac{r}{n}\right)^n = 1$$

over a full period. Since

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

the present value of $1 with continuous compounding over one period is

$$\beta = e^{-r}$$

Similarly, the present value of $1 with continuous compounding over t periods is

$$\beta^t = e^{-rt}$$

The continuous time analog of the finite horizon dynamic problem (7.7) is

$$\max_{\mathbf{a}(t)} \int_0^T e^{-rt} f(\mathbf{a}(t), s(t), t)dt + e^{-rT} v(s(T)) \tag{7.36}$$
$$\text{subject to } \dot{s} = g(\mathbf{a}(t), s(t), t)$$

given $s(0) = s_0$, with an integral replacing the sum in the objective function and a differential equation replacing the difference equation in the transition equation.

The Lagrange multipliers $(\lambda_1, \lambda_2, \dots, \lambda_T)$ in (7.8) define a functional on the set $1, 2, \dots, T$. They must be replaced in the continuous time framework by a functional $\lambda(t)$ on $[0, T]$. As in Section 7.2, it is convenient to multiply each constraint by e^{-rt} when forming the Lagrangean, to give

$$L = \int_0^T e^{-rt} f(\mathbf{a}(t), s(t), t)dt + e^{-rT} v(s(T)) - \int_0^T e^{-rt} \lambda(t)\big(\dot{s} - g(\mathbf{a}(t), s(t), t)\big)dt$$

Rearranging terms

$$\begin{aligned} L &= \int_0^T e^{-rt}\big(f(\mathbf{a}(t), s(t), t) + \lambda(t) g(\mathbf{a}(t), s(t), t)\big)dt - \int_0^T e^{-rt}\lambda(t)\dot{s}dt + e^{-rT} v(s(T)) \\ &= \int_0^T e^{-rt} H(\mathbf{a}(t), s(t), \lambda(t), t)dt - \int_0^T e^{-rt}\lambda(t)\dot{s}dt + e^{-rT} v(s(T)) \end{aligned} \tag{7.37}$$

where H is the Hamiltonian

$$H(\mathbf{a}(t), s(t), \lambda(t), t) = f(\mathbf{a}(t), s(t), t) + \lambda(t) g(\mathbf{a}(t), s(t), t)$$

Assuming for the moment that $\lambda(t)$ is differentiable, we can integrate the second term in (7.36) by parts to give

$$\int_0^T e^{-rt}\lambda(t)\dot{s}dt = e^{-rT}\lambda(T)s(T) - \lambda(0)s(0) - \int_0^T e^{-rt} s(t)\dot{\lambda}dt + r\int_0^T e^{-rt} s(t)\lambda(t)dt$$

so that the Lagrangean can be written as

$$\begin{aligned} L = \int_0^T e^{-rt}\big(H(\mathbf{a}(t), s(t), \lambda(t), t) + s(t)\dot{\lambda} - rs(t)\lambda(t)\big)dt \\ + e^{-rT} v(s(T)) - e^{-rT}\lambda(T)s(T) + \lambda(0)s(0) \end{aligned}$$

Stationarity of the Lagrangean requires

$$\begin{aligned} D_{\mathbf{a}} L &= e^{-rt} D_{\mathbf{a}} H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) = 0 \\ D_s L &= e^{-rt}\Big(D_s H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) + \dot{\lambda} - r\lambda(t)\Big) = 0 \\ D_{S(T)} L &= e^{-rT}\big(v'(s(T)) - \lambda(T)\big) = 0 \end{aligned}$$

Since $e^{-rt} > 0$ (exercise 2.6), these imply

$$\begin{aligned} D_{\mathbf{a}} H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) &= 0 \\ \dot{\lambda} &= r\lambda(t) - D_s H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) \\ \lambda(T) &= v'(s(T)) \end{aligned}$$

Of course, the optimal solution must also satisfy the transition equation

$$\dot{s} = g(\mathbf{a}(t), s(t), t)$$

More generally, the *Maximum Principle* requires that the Hamiltonian be maximized along the optimal path. Therefore the necessary conditions for an optimal solution of the continuous time problem (7.36) include

$$\begin{aligned} \mathbf{a}^*(t) &\text{ maximizes } H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) \\ \dot{s} &= D_\lambda H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) = g(\mathbf{a}(t), s(t), t) \\ \dot{\lambda} &= r\lambda(t) - D_s H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) \\ \lambda(T) &= v'(s(T)) \end{aligned}$$

Theorem 7.2 (Continuous maximum principle) *If* $\mathbf{a}(t)$ *solves the continuous finite horizon dynamic optimization problem*

$$\max_{\mathbf{a}(t)} \int_0^T e^{-rt} f(\mathbf{a}(t), s(t), t)dt + e^{-rT} v(s(T))$$
$$\textit{subject to } \dot{s} = g(\mathbf{a}(t), s(t), t)$$

given the initial state s_0*, then there exists a function* $\lambda(t)$ *such that*

$$\begin{aligned} \mathbf{a}^*(t) &\textit{ maximizes } H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) \\ \dot{s} &= D_\lambda H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) = g(\mathbf{a}(t), s(t), t) \\ \dot{\lambda} &= r\lambda(t) - D_s H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) \\ v'(s(T)) &= \lambda(T) \end{aligned}$$

where H *is the Hamiltonian*

$$H(\mathbf{a}(t), s(t), \lambda(t), t) = f(\mathbf{a}(t), s(t), t) + \lambda(t) g(\mathbf{a}(t), s(t), t) \tag{7.38}$$

Remark 7.6 (Form of the Hamiltonian) As in the discrete case (Remark 7.4), the Hamiltonian defined in equation (7.38) is known as the *current value Hamiltonian* since it measures total return at time t. Many authors present the continuous time maximum principle using the *initial value Hamiltonian* defined as

$$\tilde{H}(\mathbf{a}(t), s(t), \mu(t), t) = e^{-rt} f(\mathbf{a}(t), s(t), t) + \mu(t) g(\mathbf{a}(t), s(t), t)$$

in terms of which the necessary conditions for an optimum are

$$\begin{aligned} \mathbf{a}^*(t) &\text{ maximizes } \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) \\ \dot{s} &= D_\mu \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) = g(\mathbf{a}(t), s(t), t) \\ \dot{\mu} &= -D_s \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) \\ e^{-rT} v'(s(T)) &= \mu(T) \end{aligned} \tag{7.39}$$

While these conditions are almost identical to the corresponding conditions for the discrete time problem, there is a difference in sign between (7.39) and the corresponding condition

$$\mu_t = D_s H\big(\mathbf{a}(t), s(t), \mu(t), t\big)$$

for discrete time. The discrete time problem can be formulated in a way which is strictly analogous to the continuous time problem (see Dorfman 1969), but this formulation is a less natural extension of static optimization.

Exercise 7.12 Show that the necessary conditions for an optimum expressed in terms of the initial value Hamiltonian are

$$\begin{aligned} \mathbf{a}^*(t) &\text{ maximizes } \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) \\ \dot{s} &= D_\mu \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) = g(\mathbf{a}(t), s(t), t) \\ \dot{\mu} &= -D_s \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) \\ e^{-rT} v'(s(T)) &= \mu(T) \end{aligned}$$

Example 7.7 (Calculus of variations) The classic calculus of variations treats problems of the form

$$\max \int_0^T f(\dot{s}(t), s(t), t) dt$$

given $s(0) = s_0$. Letting $a(t) = \dot{s}(t)$, this can be cast as a standard dynamic optimization problem

$$\max_{a(t)} \int_0^T f(a(t), s(t), t) dt + v(s(T))$$
$$\text{subject to } \dot{s} = a(t)$$

given $s(0) = s_0$. The Hamiltonian is

$$H(a(t), s(t), \lambda(t), t) = f(a(t), s(t), t) + \lambda(t) a(t)$$

The necessary conditions are

$$D_a H(a(t), s(t), \lambda(t), t) = f_a(a(t), s(t), t) + \lambda(t) = 0 \tag{7.40}$$
$$\dot{s} = a(t)$$
$$\dot{\lambda} = -D_s H(a(t), s(t), \lambda(t), t) = -f_s(a(t), s(t), t) \tag{7.41}$$

Differentiating (7.40) gives

$$\dot{\lambda} = -D_t f_a(a(t), s(t), t)$$

Substituting in (7.41) and setting $a = \dot{s}$ we get

$$f_s(\dot{s}(t), s(t), t) = D_t f_{\dot{s}}(\dot{s}(t), s(t), t)$$

which is the original *Euler equation*.

As we argued in the discrete time case, an optimal solution for an infinite horizon problem must also be optimal over any finite period. It follows that the necessary conditions for the finite horizon problem (with exception of the transversality condition) are also necessary for the infinite horizon problem (Halkin 1974).

Corollary 7.2.1 (Infinite horizon continuous time) *If* $\mathbf{a}(t)$ *solves the continuous infinite horizon dynamic optimization problem*

$$\max_{\mathbf{a}(t)} \int_0^\infty e^{-rt} f(\mathbf{a}(t), s(t), t) dt$$
$$\textit{subject to } \dot{s} = g(\mathbf{a}(t), s(t), t)$$

given the initial state s_0*, then there exists a function* $\lambda(t)$ *such that*

$$\begin{aligned} \mathbf{a}^*(t) &\textit{ maximizes } H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) \\ \dot{s} &= D_\lambda H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) = g(\mathbf{a}(t), s(t), t) \\ \dot{\lambda} &= r\lambda(t) - D_s H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) \end{aligned}$$

Example 7.8 (Optimal economic growth) Formulated in continuous time, the problem of optimal economic growth is

$$\max \int_0^\infty e^{-rt} u(c(t))dt$$
$$\text{subject to } \dot{k} = F(k(t)) - c(t)$$

given $k(0) = k_0$. The Hamiltonian is

$$H(c(t), k(t), \lambda(t), t) = u(c(t)) + \lambda(t)\big(F(k(t)) - c(t)\big)$$

The necessary conditions are

$$\begin{aligned} D_c H(c(t), k(t), \lambda(t), t) &= u'(c(t)) - \lambda(t) = 0 \\ \dot{k} &= F(k(t)) - c(t) \\ \dot{\lambda} &= r\lambda(t) - D_s H(c(t), k(t), \lambda(t), t) \\ &= r\lambda(t) - \lambda(t)F'(k(t)) \\ &= (r - F'(k(t)))\lambda(t) \end{aligned} \tag{7.42}$$

Condition (7.41) implies

$$\lambda(t) = u'(c(t)) \text{ and } \dot{\lambda} = u''(c(t))\dot{c}$$

Substituting these in (7.42) gives the Euler equation

$$u''(c(t))\dot{c} = (r - F'(k(t)))u'(c(t))$$

or

$$\dot{c} = -\frac{u'(c(t))}{u''(c(t))}(F'(k(t)) - r)$$

We observe that

$$\dot{c} \gtreqless 0 \iff F'(k(t)) \gtreqless r$$

which is equivalent to the conclusion we derived using a discrete model in Example 7.4.

Exercise 7.13 (Investment in the competitive firm) A continuous time version of the investment model (Example 7.5) is

$$\max_{I(t)} \int_{t=0}^\infty e^{-rt}\big(p(t)f(k(t)) - qI(t)\big)$$
$$\text{subject to } \dot{k} = I(t) - \delta k(t)$$
$$k(0) = \bar{k}_0$$

where for simplicity we assume that k is the only input. Show that the necessary condition for an optimal investment policy is

$$p(t)f'(k(t)) = (r + \delta)q$$

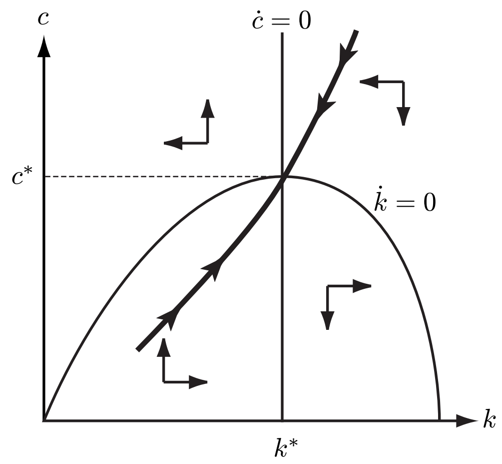

Figure 7.1: A phase diagram

7.3.1 Phase diagrams

The optimal growth example is a typical stationary dynamic optimization problem, in which the functions f and g are independent of time, which enters only through the discount factor e^{-rt}. As in the discrete case, it may be reasonable to assume that the system converges to a steady state. In simple cases, convergence to the steady state can be analyzed using a *phase diagram*. We illustrate by means of an example.

The dynamics of the optimal economic growth problem are described by the following pair of differential equations

$$\dot{c} = -\frac{u'(c(t))}{u''(c(t))}(F'(k(t)) - r) \tag{7.43}$$

$$\dot{k} = F(k(t)) - c(t) \tag{7.44}$$

A solution of (7.43) and (7.44) is a pair of functions $c(t)$ and $k(t)$, each of which can be represented by a path or trajectory in (k, c) space (see Figure 7.1). A steady state requires both $\dot{c} = 0$ and $\dot{k} = 0$. Each condition determines a locus in (k, c) space, with the steady state at their intersection. These loci divide the space into four regions, in each of which the paths of c and k have different directions. A unique path or trajectory passes through every point. In particular, there is a unique path passing through the steady state equilibrium. By analysing the direction of flow in each region, we observe that this path leads towards the steady state from the two regions through which it passes. However, any deviation from this steady state path leads away from the equilibrium. We conclude that for each initial capital stock k_0, there is a unique optimal path leading to the steady state, while all other paths eventually lead away from the steady state. Thus the steady state equilibrium is a saddle point. This unique optimal path can be attained by chosing the appropriate level of initial consumption and thereafter following the optimal path determined by (7.43) and (7.44).

Example 7.9 (Optimal investment with adjustment costs) A shortcoming of the investment model of example 7.5 and exercise 7.13 is that the cost of investment is assumed to be linear, which precludes adjustment costs. A more realistic model allows for the cost of investment to be convex, so that the dynamic optimization problem is

$$\int_{t=0}^{\infty} e^{-rt}\big(p(t)f(k(t)) - c(I(t))\big)$$
$$\dot{k} = I(t) - \delta k(t)$$
$$k(0) = \bar{k}_0$$

with $c''(I(t)) > 0$. The Hamiltonian is

$$H = e^{-rt}\big(p(t)f(k(t)) - c(I(t))\big) + \lambda(t)\big(I(t) - \delta k(t)\big)$$

The first-order conditions are

$$H_I = -e^{-rt}c'(I(t)) + \lambda(t) = 0 \tag{7.45}$$
$$\dot{k} = I(t) - \delta k(t)$$
$$\dot{\lambda} = -H_k = -e^{-rt}p(t)f'(k(t)) + \delta\lambda(t) \tag{7.46}$$

Equation (7.45) implies

$$\lambda(t) = e^{-rt}c'(I(t)) \tag{7.47}$$

or

$$\mu(t) = e^{rt}\lambda(t) = c'(I(t)) \tag{7.48}$$

$\mu(t)$ is the current value multiplier, the shadow price of capital. It can be shown that

$$\mu(t) = \int_t^{\infty} e^{-r(s-t)}e^{-\delta(s-t)}p(t)f'(k(t))dt = \int_t^{\infty} e^{-(r+\delta)(s-t)}p(t)f'(k(t))dt \tag{7.49}$$

which is the present value of the total additional revenue (marginal revenue product) accruing to the firm from an additional unit of investment, allowing for depreciation. Equation (7.49) states simply that, at each point of time, investment is taken to the point at which the marginal value of investment is equal to its marginal cost. Equations (7.48) and (7.49) together with the transition equation determine the evolution of the capital stock, but it is impossible to obtain a closed form solution without further specification of the model. Instead, it is more tractable to resort to study the qualitative nature of a solution using a phase diagram.

Differentiating (7.47)

$$\dot{\lambda} = e^{-rt}c''(I)\dot{I} - re^{-rt}c'(I)$$

Substituting into (7.49) and using (7.48) yields

$$\begin{aligned} e^{-rt}c''(I(t))\dot{I} - re^{-rt}c'(I(t)) &= -e^{-rt}p(t)f'(k(t)) + \delta\lambda(t) \\ &= -e^{-rt}p(t)f'(k(t)) + \delta e^{-rt}c'(I(t)) \end{aligned}$$

Cancelling the common terms and rearranging

$$\dot{I} = \frac{(r+\delta)c'(I(t)) - p(t)f'(k(t))}{c''(I(t))}$$

The optimal policy is characterised by a pair of differential equations

$$\dot{I} = \frac{(r+\delta)c'(I(t)) - p(t)f'(k(t))}{c''(I(t))}$$
$$\dot{k} = I(t) - \delta k(t)$$

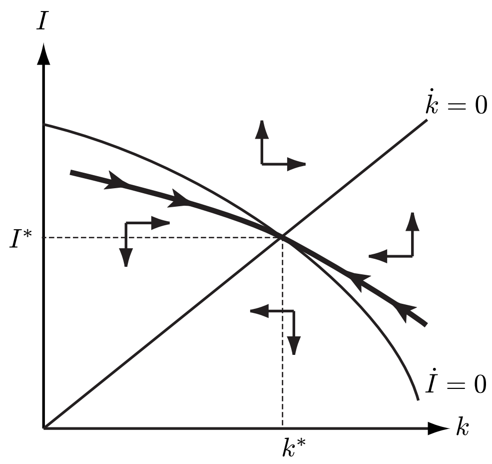

Figure 7.2: Optimal investment with adjustment costs

In the steady state solution (I^*, k^*)

$$\dot{k} = 0 \implies I^* = \delta k^*$$
$$\dot{I} = 0 \implies p(t)f'(k^*) = (r+\delta)c'(I^*)$$

In the steady state, there is no net investment and the capital stock k^* is determined where the marginal benefit of further investment is equal to the marginal cost. The steady state equilibrium is a saddlepoint, with a unique optimal path to the equilibrium from any initial state (Figure 7.2).

Exercise 7.14 (Dynamic limit pricing) Consider a market where there is a dominant firm and a competitive fringe. The residual demand facing the dominant firm is

$$f(p(t)) = a - x(t) - bp(t)$$

where $x(t)$ is the output of the competitive fringe. Entry and exit of fringe firms depends upon the price set by the dominant firm. Specifically

$$\dot{x}(t) = k(p(t) - \bar{p})$$

We can think of $\bar{p}$ as being the marginal cost of the competitive fringe. For simplicity, we assume that the dominant firm has zero marginal cost, so that $\bar{p}$ is its cost advantage.

If the dominant firm exploits its market power by pricing above the "limit price" $\bar{p}$, it increases current profits at the expense of market share. Investigate the optimal pricing policy to maximize the discounted present value of profits

$$\int_0^\infty e^{-rt} p(t)(a - x(t) - bp(t))dt$$

where r is the rate of interest. What happens to the market share of the dominant firm in the long run?

7.4 Dynamic programming

Dynamic programming is an alternative approach to dynamic optimization which facilitates incorporation of uncertainty and lends itself to electronic computation. Again consider the general dynamic optimization problem (7.7)

$$\max_{\mathbf{a}_t \in A_t} \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, \mathbf{s}_t) + \beta^T v_T(\mathbf{s}_T) \tag{7.50}$$
$$\text{subject to } s_{t+1} = \mathbf{g}_t(\mathbf{a}_t, \mathbf{s}_t), \quad t = 0, \dots, T-1$$

where we have added a time subscript to the value of the terminal state v. The (maximum) value function for this problem is

$$v_0(s_0) = \max_{\mathbf{a}_t \in A_t} \left\{ \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T v_T(s_T) : s_{t+1} = g_t(a_t, s_t),\ t = 0, 1, \dots, T-1 \right\}$$

By analogy, we define the value function for intermediate periods

$$v_t(s_t) = \max_{\mathbf{a}_t \in A_t} \left\{ \sum_{\tau=t}^{T-1} \beta^{\tau - t} f_\tau(a_\tau, s_\tau) + \beta^T v_T(s_T) : s_{\tau+1} = g_\tau(a_\tau, s_\tau),\ \tau = t, t+1, \dots, T-1 \right\}$$

The value function measures the best that can be done given the current state and remaining time. It is clear that

$$\begin{aligned} v_t(s_t) &= \max_{\mathbf{a}_t \in A_t} \{ f_t(a_t, s_t) + \beta v_{t+1}(s_{t+1}) : s_{t+1} = g_t(a_t, s_t) \} \\ &= \max_{\mathbf{a}_t \in A_t} \{ f_t(a_t, s_t) + \beta v_{t+1}\big(g_t(a_t, s_t)\big) \} \end{aligned} \tag{7.51}$$

This fundamental recurrence relation, which is known as *Bellman's equation*, makes explicit the tradeoff between present and future values. It embodies the *principle of optimality*:

> An optimal policy has the property that, whatever the intial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision (Bellman, 1957).

The principle of optimality asserts *time-consistency* of the optimal policy.

Assuming v_{t+1} is differentiable and letting

$$\lambda_{t+1} = v'_{t+1}(s_{t+1})$$

the first-order condition for the maximization in Bellman's equation (7.51) is

$$D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta \lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) = 0, \quad t = 0, 1, \dots, T-1 \tag{7.52}$$

which is precisely the Euler equation (7.11) derived using the Lagrangean approach. Moreover, the derivative of the value function $\lambda_t = v'_t(s_t)$ follows an analogous recursion which can be shown as follows. Let $a_t = h_t(s_t)$ define the *policy function*, the solution of the maximization in (7.52). Then

$$v(s_t) = f_t(h_t(s_t), s_t) + \beta v_{t+1}(g_t(h_t(s_t), s_t))$$

Assuming h and v are differentiable (and suppressing function arguments for clarity)

$$\begin{aligned} \lambda_t = v'_t(s_t) &= D_a f_t D_s h_t + D_s f_t + \beta \lambda_{t+1}(D_a g_t D_s h_t + D_s g_t) \\ &= D_s f_t + \beta \lambda_{t+1} D_s g_t + (D_a f_t + \beta \lambda_{t+1} D_a g_t) D_s h_t \end{aligned}$$

where $\lambda_{t+1} = v'(s_{t+1})$. But the term in brackets is zero (by the first-order condition (7.52)) and therefore

$$\lambda_t = D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta \lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t), \quad t = 1, 2, \dots, T-1$$

which is precisely the recursion (7.12) we previously derived using the Lagrangean technique. Coupled with the transition equation and the boundary conditions, the optimal policy is characterised by

$$D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) = 0, \qquad t = 0, 1, \ldots, T-1 \tag{7.53}$$

$$D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) = \lambda_t, \qquad t = 1, 2, \ldots, T-1 \tag{7.54}$$

$$s_{t+1} = g_t(\mathbf{a}_t, s_t), \quad t = 0, 1, \ldots, T-1 \tag{7.55}$$

$$\lambda_T = v'(s_T) \tag{7.56}$$

This should not be surprising. Indeed, it would be disturbing if, on the contrary, our characterization of an optimal solution varied with the method adopted. The main attraction of dynamic programming is that it offers an alternative method of computation, *backward induction*, which is particularly amenable to programmable computers. This is illustrated in the following example.

Example 7.10 (Closing the mine) Consider again the question posed in example 7.1. Suppose you own a mine. Your mining licence will expire in three years and will not be renewed. There are known to be 128 tons of ore remaining in the mine. The price is fixed at \$1 a ton. The cost of extraction is q_t^2/x_t where q_t is the rate of extraction and x_t is the stock of ore. Ignoring discounting for simplicity ($\beta = 1$), the optimal production plan solves

$$\max_{q_t, x_t} \sum_{t=0}^{3} \left(1 - \frac{q_t}{x_t}\right) q_t$$
$$\text{subject to } x_{t+1} = x_t - q_t, \quad t = 0, 1, 2$$

Previously (example 7.1) we solved this problem using the Lagrangean approach. Here, we solve the same problem using dynamic programming and backward induction.

First, we observe that $v_3(x_3) = 0$ by assumption. Therefore

$$\begin{aligned} v_2(x_2) &= \max_q \left\{ \left(1 - \frac{q}{x_2}\right) q + v_3(x_3) \right\} \\ &= \max_q \left(1 - \frac{q}{x_2}\right) q \end{aligned}$$

which is maximized when $q = x_2/2$ giving

$$v_2(x_2) = \left(1 - \frac{x_2}{2}\frac{1}{x_2}\right)\frac{x_2}{2} = \frac{x_2}{4}$$

Therefore

$$\begin{aligned} v_1(x_1) &= \max_q \left\{ \left(1 - \frac{q}{x_1}\right) q + v_2(x_2) \right\} \\ &= \max_q \left\{ \left(1 - \frac{q}{x_1}\right) q + \frac{1}{4}(x_1 - q) \right\} \end{aligned}$$

The first-order condition is

$$1 - 2\frac{q}{x_1} - \frac{1}{4} = 0$$

which is satisfied when $q = 3x_1/8$ so that

$$v_1(x_1) = \left(1 - \frac{3x_1}{8x_1}\right)\frac{3x_1}{8} + \frac{5x_1}{32} = \frac{15}{64}x_1 + \frac{5}{32}x_1 = \frac{25}{64}x_1$$

In turn

$$v_0(x_0) = \max_q \left\{ \left(1 - \frac{q}{x_0}\right) q + v_1(x_1) \right\}$$
$$= \max_q \left\{ \left(1 - \frac{q}{x_0}\right) q + \frac{25}{64}(x_0 - q) \right\}$$

The first-order condition is

$$1 - 2\frac{q}{x_0} - \frac{25}{64} = 0$$

which is satisfied when $q = 39x_0/128$. The optimal policy is

t	x_t	q_t
0	128	39
1	89	$\frac{3}{8}89 = 33.375$
2	$\frac{5}{8}89 = 55.625$	$\frac{5}{16}89 = 27.8125$

In summary, we solve the problem by computing the value function starting from the terminal state, in the process of which we compute the optimal solution.

Typically, economic models are not solved for a specific solutions, but general solutions are characterized by relationships such as the Euler equation (7.20). In such cases, the dynamic programming approach often provides a more elegant derivation of the basic Euler equation characterising the optimal solution than does the Lagrangean approach, although the latter is more easily related to static optimization. This is illustrated in the following example.

Example 7.11 (Optimal economic growth) Consider again the optimal economic growth model (example 7.2)

$$\max_{c_t} \sum_{t=0}^{T-1} \beta^t u(c_t) + \beta^T v(k_T)$$
$$\text{subject to } k_{t+1} = F(k_t) - c_t$$

where c is consumption, k is capital, $F(k_t)$ is the total supply of goods available at the end of period t, comprising current output plus undepreciated capital, and $v(k_T)$ is the value of the remaining capital at the end of the planning horizon.

Bellman's equation is

$$v_t(w_t) = \max_{c_t}\{u(c_t) + \beta v_{t+1}(k_{t+1})\}$$
$$= \max_{c_t}\{u(c_t) + \beta v_{t+1}\big(F(k_t) - c_t\big)\}$$

The first-order condition for this problem is

$$u'(c_t) - \beta v'_{t+1}\big(F(k_t) - c_t\big) = 0 \tag{7.57}$$

But

$$v_{t+1}(k_{t+1}) = \max_{c_{t+1}}\{u(c_{t+1}) + \beta v_{t+2}\big(F(k_{t+1} - c_{t+1})\big)\} \tag{7.58}$$

By the envelope theorem (theorem 6.2)

$$v'_{t+1}(k_{t+1}) = \beta v'_{t+2}\big(F(k_{t+1}) - c_{t+1}\big)F'(k_{t+1}) \tag{7.59}$$

The first-order condition for (7.58) is

$$u'(c_{t+1}) - \beta v'_{t+2}\big(F(k_{t+1}) - c_{t+1}\big) = 0$$

Substituting in (7.59) gives

$$v'_{t+1}(k_{t+1}) = u'(c_{t+1})F'(k_{t+1})$$

Substituting the latter in (7.57) gives the Euler equation (7.20)

$$u'(c_t) = \beta u'(c_{t+1})F'(k_{t+1})$$

Note how this derivation of (7.20) is simpler and more elegant than the corresponding analysis in example 7.2.

Exercise 7.15 (Optimal savings) In the optimal saving model discussed in Section 1

$$\max \sum_{t=0}^{T-1} \beta^t u(c_t)$$

$$\text{subject to } w_{t+1} = (1+r)(w_t - c_t), \quad t - 0, 1, \ldots, T-1$$

use dynamic programming to derive (7.5).

7.4.1 Infinite horizon

In the stationary infinite horizon problem

$$\max \sum_{t=0}^{\infty} \beta^t f(a_t, s_t)$$

$$\text{subject to } s_{t+1} = g(a_t, s_t), \quad t = 0, 1, \ldots,$$

the value function

$$v(s_0) = \max \left\{ \sum_{t=0}^{\infty} \beta^t f(a_t, s_t) : s_{t+1} = g(a_t, s_t),\ t = 0, 1, \ldots \right\}$$

is also stationary (independent of t) . That is, the value function is common to all time periods, although of course its value $v(s_t)$ will vary with s_t. Bellman's equation

$$v(s_t) = \max_{a_t}\{f(a_t, s_t) + v(g(a_t, s_t))\}$$

must hold in all periods and all states, so we can dispense with the subscripts

$$v(s) = \max_{a}\{f(a, s) + \beta v(g(a, s))\} \tag{7.60}$$

The first-order conditions for (7.60) can be used to derive the Euler equation to characterize the optimal solution, as we did in Example 7.11.

In many economic models, it is possible to dispense with the separate transition equation by identifying the control variable in period t with the state variable in the subsequent period. For example, in the economic growth model, we can consider the choice in each period effectively as given capital stock today, select capital stock tomorrow, with consumption being determined as the residual. Letting x_t denote the decision variable, the optimization problem becomes

$$\max_{x_1, x_2, \ldots} \sum_{t=0}^{\infty} \beta^t f(x_t, x_{t+1})$$

$$\text{subject to } x_{t+1} \in G(x_t),\ t = 0, 1, 2, \ldots$$

$$x_0 \in X \text{ given}$$

This was the approach we took in example 2.32. Bellman's equation for this problem is

$$v(x) = \max_y \{f(x, y) + \beta v(y)\} \tag{7.61}$$

This formulation enables an especially elegant derivation of the Euler equation. The first-order condition for the maximum in (7.61) is

$$f_y + \beta v'(y) = 0$$

Using the envelope theorem (theorem 6.2)

$$v'(y) = f_x$$

Substituting, the first-order condition becomes

$$f_y + \beta f_x = 0$$

Example 7.12 (Optimal economic growth) Substituting for c_t using the transition equation

$$c_t = F(k_t) - k_{t+1}$$

the optimal growth problem (example 7.11) can be expressed as

$$\max \sum_{t=0}^{\infty} \beta^t u\big(F(k_t) - k_{t+1}\big)$$

Bellman's equation is

$$v(k_t) = \max_{k_{t+1}} \{u(F(k_t) - y) + \beta v(k_{t+1})\}$$

The first-order condition for a maximum is

$$-u'(c_t) + \beta v'(k_{t+1}) = 0$$

where $c(t) = F(k_t) - k_{t+1}$. Applying the envelope theorem

$$v'(k_t) = u'(c_t)F'(k_t)$$

and therefore

$$v'(k_{t+1}) = u'(c_{t+1})F'(k_{t+1})$$

Substituting in the first-order condition, we derive the Euler equation

$$u'(c_t) = \beta u'(c_{t+1})F'(k_{t+1})$$

For a stationary infinite horizon problem, Bellman's equation (7.60) or (7.61) defines a *functional equation*, an equation in which the variable is the function v. From another perspective, Bellman's equation defines an operator $v \to v$ on the space of value functions. Under appropriate conditions, this operator has a unique fixed point, which is the unique solution of the functional equation (7.61). On this basis, we can guarantee the existence and uniqueness of the optimal solution to an infinite horizon problem, and also deduce some of the properties of the optimal solution (exercises 2.125, 2.126 and 3.158 and examples 2.93 and 3.64).

In those cases in which we want to go beyond the Euler equation and these deducible properties to obtain an explicit solution, we need to find the solution v of this functional equation. Given v, it is straightforward to solve (7.60) or (7.61) successively to compute the optimal policy. How do we solve the functional equation? Backward induction, which we used in example 7.10, is obviously precluded with the infinite horizon. There are at least three practical approaches to solving Bellman's equation in infinite horizon problems

- informed guess and verify
- value function iteration
- policy function iteration (Howard improvement algorithm)

In simple cases, it may be possible to guess the functional form of the value function, and then verify that it satisfies Bellman's equation. Given that Bellman's equation has a unique solution, we can be confident that our verified guess is the only possible solution. In other cases, we can proceed by successive approximation. Given a particular value function v^1, (7.61) defines another value function v^2 by

$$v^2(s) = \max_a \{f(a, s) + \beta v^1(g(a, s))\} \tag{7.62}$$

and so on. Eventually, this iteration converges to the unique solution of (7.62). Policy function iteration starts with a feasible policy $h^1(s)$ and computes the value function assuming that policy is applied consistently

$$v^1(s) = \max \sum_{t=0}^{\infty} \beta^t f(h^1(s_t), s_t) \text{ subject to } s_{t+1} = g(a_t, s_t), \quad t = 0, 1, \ldots,$$

Given this approximation to the value function, we compute a new policy function h^2 which solves Bellman's equation assuming this value function, that is

$$h^2(s) = \arg\max_a \{f(a, s) + \beta v^1(g(a, s))\}$$

and then use this policy function to define a new value function v^2. Under appropriate conditions, this iteration will converge to the optimal policy function and corresponding value function. In many cases, convergence is faster than mere value function iteration (Ljungqvist and Sargent 2000: 33).

Example 7.13 (Optimal economic growth) In the optimal economic growth model (example 7.11, assume that utility is logarithmic and the technology Cobb-Douglas, so that the optimization problem is

$$\max_{c_t} \sum_{t=0}^{T-1} \beta^t \log(c_t) + v(k_T)$$
$$\text{subject to } k_{t+1} = Ak_t^\alpha - c_t$$

with $A > 0$ and $0 < a < 1$.

Starting with an initial value function

$$v^1(k) = 0$$

the first iteration implies an optimal consumption level of Ak^α and a second value function of

$$v^1(k) = \log A + \alpha \log(k)$$

Continuing in this fashion, we find that the iterations converge to

$$v(k) = C + D\log(k)$$

with

$$C = \frac{1}{1-\beta}\left(\log(A - \alpha\beta A) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta A)\right) \text{ and } D = \frac{\alpha}{1-\alpha\beta}$$

Exercise 7.16 Verify that the iteration described in the previous example converges to

$$v(k) = C + D\log(k)$$

with

$$C = \frac{1}{1-\beta}\left(\log(A - \alpha\beta A) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta A)\right) \text{ and } D = \frac{\alpha}{1-\alpha\beta}$$

Exercise 7.17 Suppose that we (correctly) the conjecture that the value function takes the form

$$v(k) = C + D\log(k)$$

with undetermined coefficients C and D. Verify that this satisfies Bellman's equation with

$$C = \frac{1}{1-\beta}\left(\log(A - \alpha\beta A) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta A)\right) \text{ and } D = \frac{\alpha}{1-\alpha\beta}$$

7.5 Notes

Dixit (1990) gives a nice intuitive introduction to dynamic optimization in economics, emphasizing the parallel with static optimization. Another introductory treatment, focusing on resource economics, can be found in Conrad and Clark(1987).

Many texts aimed at economists follow the historical mathematical development, starting with the calculus of variations and then proceeding to optimal control theory. Examples include Chiang (1992), Kamien and Schwartz (1991) and Hadley and Kemp (1971), listed in increasing level of difficulty. The problem with this approach is that it requires sustained effort to reach the modern theory. A useful exception to this traditional structure is Leonard and Long (1992), which starts with the maximum principle, after reviewing static optimization and differential equations. They also provide a good discussion of the use of phase diagrams in analysing dynamic models.

Leading texts presenting the dynamic optimizing approach to macroeconomics include Blanchard and Fischer (1989), Ljungqvist and Sargent (2000), Sargent (1987) and Stokey and Lucas (1989). Our discussion of the Howard policy improvement algorithm is based on Ljungqvist and Sargent (2000)

Example 7.1 is adapted from Conrad and Clark (1987) Exercise 7.9 is based on Stiglitz (1976). Exercise 7.11 is adapted from Tirole (1988). The continuous time version of the optimal economic growth model (Example 7.8), known as the Ramsey model, is the prototype for the study of growth and intertemporarl allocation (see Blanchard and Fischer 1989). Exercise 7.14 is adapted from Gaskins (1971).

7.6 References

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Solutions to exercises

7.1 With $u(c) = \log c$, $u'(c) = 1/c$. Substituting in (7.2)

$$\frac{c_1}{\beta c_0} = 1 + r$$

or

$$c_1 = \beta(1 + r)c_0$$

Substituting in the budget constraint

$$\beta(1 + r)c_0 = (1 + r)(w - c_0)$$

Solving for c_1 and c_2 gives

$$c_0 = \frac{w}{1 + \beta}, \qquad c_1 = (1 + r)\frac{\beta w}{1 + \beta}$$

7.2 The first-order condition is

$$\text{Intertemporal MRS} = \frac{u'(c_0)}{\beta u'(c_1)} = \frac{1}{\beta}\sqrt{\frac{c_1}{c_0}} = 1 + r$$

which implies that

$$c_0 = \alpha w, \qquad c_1 = (1 - \alpha)w$$

where

$$\alpha = \frac{1}{1 + \beta^2(1 + r)}$$

so that

$$D_r c_0 < 0$$

7.3 Let w_t denote the wealth remaining at the beginning of period t. The consumer should consume all remaining wealth in period $T - 1$ so that

$$\begin{aligned} c_{T-1} = w_{T-1} &= (1 + r)(w_{T-2} - c_{T-2}) \\ &= (1 + r)w_{T-2} - (1 + r)c_{T-2} \\ &= (1 + r)^2(w_{T-3} - c_{T-3}) - (1 + r)c_{T-2} \\ &= (1 + r)^2 w_{T-3} - (1 + r)^2 c_{T-3} - (1 + r)c_{T-2} \\ &= (1 + r)^{T-1}w_0 - (1 + r)^{T-1}c_0 - \cdots - (1 + r)^2 c_{T-3} - (1 + r)c_{T-2} \end{aligned}$$

which can be rewritten as

$$(1 + r)^{T-1}c_0 + \cdots + (1 + r)^2 c_{T-3} + (1 + r)c_{T-2} + c_{T-1} = (1 + r)^{T-1}w_0$$

or

$$(1 + r)^T c_0 + \cdots + (1 + r)^3 c_{T-3} + (1 + r)^2 c_{T-2} + (1 + r)c_{T-1} = (1 + r)^T w_0$$

or

$$c_0 + \frac{1}{1 + r}c_1 + \cdots + \frac{1}{(1 + r)^{T-1}}c_{T-1} = w_0$$

The consumer's problem is

$$\max_{c_t} \sum_{t=0}^{T-1} \beta^t u(c_t)$$
$$\text{subject to } (1+r)^T c_0 + \cdots + (1+r)^2 c_{T-2} + (1+r)c_{T-1} = (1+r)^T w_0$$

The Lagrangean for this problem is

$$L = \sum_{t=0}^{T-1} \beta^t u(c_t) - \lambda\big((1+r)^T c_0 + \cdots + (1+r)^2 c_{T-2} + (1+r)c_{T-1} = (1+r)^T w_0\big)$$

The first-order conditions are

$$D_{c_t} L = \beta^t u'(c_t) - \lambda(1+r)^{T-t} = 0, \; t = 0, 1, \ldots, T-1$$

which imply

$$\beta^t u'(c_t) = \lambda(1+r)^{T-(t+1)}(1+r) = \beta^{t+1} u'(c_{t+1})(1+r)$$

or

$$u'(c_t) = \beta u'(c_{t+1})(1+r), \; t = 0, 1, \ldots, T-1$$

which is the same intertemporal allocation condition (7.5) obtained using separate constraints for each period.

7.4 Setting $\mathbf{a}_t = c_t$, $s_t = w_t$, $f(\mathbf{a}_t, s_t) = u(c_t)$, $g(\mathbf{a}_t, s_t) = (1+r)(w_t - c_t)$, and $v(s_T) = 0$, the optimality conditions (7.11) to (7.14) are

$$\begin{aligned} &u'(c_t) - \beta\lambda_{t+1}(1+r) = 0 \\ &\beta\lambda_{t+1}(1+r) = \lambda_t \\ &w_{t+1} = (1+r)(w_t - c_t) \\ &\lambda_T = 0 \end{aligned}$$

which together imply

$$u'(c_t) = \beta u'(c_{t+1})(1+r), \; t = 0, 1, \ldots, T-1$$

as required.

7.5 Setting $\mathbf{a}_t = c_t$, $s_t = k_t$, $f(a_t, s_t) = u(c_t)$, $g(a_t, s_t) = F(k_t) - c_t$ and using μ to denote the Lagrange multipliers, the necessary conditions (7.21) to (7.24) are

$$\begin{aligned} \beta^t u'(c_t) - \mu_{t+1} = 0 & \qquad t = 0, 1, \ldots, T-1 \\ \mu_{t+1} F'(k_t) = \mu_t & \qquad t = 1, 2, \ldots, T-1 \\ k_{t+1} = F(k_t) - c_t & \qquad t = 0, 1, \ldots, T-1 \\ \mu_T = v'(k_T) & \end{aligned}$$

Substituting $\mu_t = \beta^t \lambda_t$ and $\mu_{t+1} = \beta^{t+1}\lambda_{t+1}$ gives

$$\begin{aligned} \beta^t u'(c_t) - \beta^{t+1}\lambda_{t+1} = 0 & \qquad t = 0, 1, \ldots, T-1 \\ \beta^{t+1}\lambda_{t+1} F'(k_t) = \beta^t \lambda_t & \qquad t = 1, 2, \ldots, T-1 \\ k_{t+1} = F(k_t) - c_t & \qquad t = 0, 1, \ldots, T-1 \\ \mu_T = v'(k_T) & \end{aligned}$$

or

$$
\begin{aligned}
u'(c_t) = \beta\lambda_{t+1} &= 0 && t = 0, 1, \ldots, T-1 \\
\lambda_t &= \beta\lambda_{t+1}F'(k_t) && t = 1, 2, \ldots, T-1 \\
k_{t+1} &= F(k_t) - c_t && t = 0, 1, \ldots, T-1 \\
\mu_T &= v'(k_T)
\end{aligned}
$$

which are the same as conditions (7.15) to (7.18).

7.6 Assigning multiplier μ to the terminal constraint

$$h(s_T) = \bar{s} - s_T \square 0$$

the Lagrangean for this problem is

$$L = \sum_{t=0}^{T-1} \beta^t f_t(\mathbf{a}_t, s_t) - \sum_{t=0}^{T-1} \beta^{t+1}\lambda_{t+1}\big(s_{t+1} - g_t(\mathbf{a}_t, s_t)\big) - \beta^T \mu(\bar{s} - s_T)$$

which can be rewritten as

$$
\begin{aligned}
L = {} & f_0(\mathbf{a}_0, s_0) + \beta\lambda_1 g_0(\mathbf{a}_0, s_0) \\
& + \sum_{t=1}^{T-1} \beta^t \Big(f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} g_t(\mathbf{a}_t, s_t) - \lambda_t s_t \Big) \\
& - \beta^T \lambda_T s_T - \beta^T \mu(\bar{s} - s_T)
\end{aligned}
$$

A necessary condition for optimality is the existence of multipliers $\lambda_1, \lambda_2, \ldots, \lambda_T$ such that the Lagrangean is stationary, that is for $t = 0, 1, \ldots, T-1$

$$D_{\mathbf{a}_t} L = \beta^t \big(D_{\mathbf{a}_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{\mathbf{a}_t} g_t(\mathbf{a}_t, s_t) \big) = 0$$

Similarly, in periods $t = 1, 2, \ldots, T-1$, the resulting s_t must satisfy

$$D_{s_t} L = \beta^t \big(D_{s_t} f_t(\mathbf{a}_t, s_t) + \beta\lambda_{t+1} D_{s_t} g_t(\mathbf{a}_t, s_t) - \lambda_t \big) = 0$$

as well as the transition equations

$$s_{t+1} = g_t(\mathbf{a}_t, s_t), \quad t = 0, \ldots, T-1$$

The equations imply (7.11) to (7.13). The terminal state s_T must satisfy

$$D_{s_T} L = \beta^T \big(-\lambda_T + \mu \big) = 0$$

with $\mu \geq 0$ and $\mu(\bar{s} - s_T) = 0$. This implies that $\lambda_T = \mu \geq 0$ and therefore we have

$$\lambda_T \geq 0 \text{ and } \lambda_T(\bar{s} - s_T) = 0$$

As in theorem 7.1, these conditions are also sufficient.

7.7 Necessity of the optimality conditions follows from Corollary 5.2.1. As in Theorem 7.1, the necessary conditions imply that $\lambda_t \geq 0$ for every t. Therefore the Lagrangean is concave, so that stationarity is sufficient for a global optimum (Exercise 5.20).

7.8 Let T' denote the period in which the resource is exhausted, that is $x_{T'} = 0$ while $x_t > 0$ for all $t < T'$. This implies that $\lambda_{t+1} = \lambda_t$ for all $t < T$. That is, $\lambda_t = \lambda_{T'}$ constant for $t = 0, 1, \ldots, T'-1$. The periods are of two types.

Productive periods ($q_t > 0$) In productive periods, the allocation of extraction is arranged so that the discounted marginal profit is equal in all periods, that is

$$\beta^t\big(m_t(q_t) - c_t'(q_t) = \lambda_{T'}$$

Nonproductive periods ($q_t = 0$) In nonproductive periods, nothing is extracted, since the marginal profit of the first unit is less than its opportunity cost $\lambda_{T'}$.

$$\beta^t m_t(q_t) \Box \lambda_{T'}$$

7.9 In a competitive industry with zero extraction costs, Hotelling's rule implies that the price rises at the rate of interest, that is

$$\frac{p_{t+1}}{p_t} = 1 + r \tag{7.63}$$

Otherwise, there are opportunities for profitable arbitrage. To compare with the implicit rate of price change under monopoly, we note that marginal revenue can be rewritten as

$$m_t = p_t(q_t) + p'(q_t)q_t = p_t(1 + p'(q_t)\frac{q_t}{p_t}) = p_t(1 + \frac{1}{\epsilon_t})$$

where ϵ_t is the elasticity of demand in period t. Substituting in (7.28), the price under monopoly evolves according to the equation

$$\frac{p_{t+1}\left(1 + \frac{1}{\epsilon_{t+1}}\right)}{p_t\left(1 + \frac{1}{\epsilon_t}\right)} = 1 + r$$

or

$$\frac{p_{t+1}}{p_t} = (1 + r)\left(\frac{1 + \frac{1}{\epsilon_t}}{1 + \frac{1}{\epsilon_{t+1}}}\right) \tag{7.64}$$

Comparing (7.63) and (7.64), we conclude that in a monopoly the price will rise faster (slower) than the rate of interest if the elasticity of demand ($|\epsilon|$) is increasing (decreasing). This implies that a monopoly will extract an exhaustible resource at a slower (faster) rate than a competitive industry if the elasticity of demand increases (decreases) over time.

Increasing elasticity is likely if substitutes develop over time. Therefore, market concentration is likely to impart a conservative bias to the extraction of an exhaustible resource. The basic insight of this problem is that the monopolists, like the competitor, will eventually exhaust the resource. The monopolist cannot profit by restricting total output, as in the case of a produced commodity. They can only exploit market power by rearranging the pattern of sales over time.

Contrary to the popular belief that a monopoly will rapidly deplete an exhaustible resource, analysis suggests that monopolists may be more conservationist than a competitive market. As we showed above, this will be the case if demand elasticity increases over time, as might be expected as substitutes become available. Extraction costs may also impart a conservationist bias to the monopoly.

7.10 If investment is irreversible ($I_t \geq 0$), the firm's problem is

$$\max \sum_{t=0}^{\infty} \delta^t \pi_t = \sum_{t=0}^{\infty} \delta^t \big(p_t f(k_t, l_t) - w_t l_t - qI_t\big)$$

subject to $I_t \geq 0$

$$k_0 = \bar{k}_0, \qquad k_{t+1} = (1-\rho)k_t + I_t, \qquad t = 0, 1, 2, \dots$$

Note that we also require $k_t \geq 0$ and $l_t \geq 0$ but these constraints are presumably not binding in the optimal solution and can be ignored in the analysis. However, the nonnegativity constraint $I_t \geq 0$ is quite possibly binding and an interior solution cannot be guaranteed. The necessary conditions become

$$\begin{aligned} H_l &= p_t f_l(k_t, l_t) - w_t = 0 \\ &\max_{I \geq 0} H \\ k_{t+1} &= (1-\rho)k_t + I_t \\ \lambda_t &= f_k(k_t, l_t) + \delta\lambda_{t+1}(1-\rho) \end{aligned}$$

Maximising the Hamiltonian with respect to I (7.64) requires

$$H_I = -q + \delta\lambda_{t+1} \square 0 \qquad I_t \geq 0 \text{ and } I_t(q - \delta\lambda_{t+1}) = 0$$

In other words, the $\delta\lambda_{t+1} \square q$ in every period and the firm invests $I_t > 0$ if and only if $\delta\lambda_{t+1} = q$

As in the previous question, optimality requires adjusting labour in each period so that

$$p_t f_l(k_t, l_t) = w_t$$

The necessary conditions for capital accumulation are a little more complicated. Assume $I_{t-1} > 0$. Then $\delta\lambda_t = q$ so that $\lambda_t = q/\delta$. Substituting in (7.63) and using (7.64)

$$\begin{aligned} \frac{q}{\delta} &= f_k(k_t, l_t) + \delta\lambda_{t+1}(1-\rho) \\ &\square f_k(k_t, l_t) + q(1-\rho) \end{aligned}$$

which implies

$$f_k(k_t, l_t) \geq (r+q)q$$

with

$$f_k(k_t, l_t) = (r+q)q \iff I_t > 0$$

7.11 (a) A competitive recycling industry will produce where price equals marginal cost, that is $p_t = C'(x_t)$. Since C is assumed to be strictly convex, C' has an inverse x such that

$$x_t = x(p_t)$$

By the inverse function theorem

$$x' = \frac{1}{C''} > 0$$

That is, x is increasing in p.

(b) The monopolist's optimization problem is

$$\max \sum_{t=0}^{\infty} \beta^t (P(q_t) - c) y_t$$
$$\text{where } q_t = y_t - x_t q_{t-1}$$

From the constraint $y_t = q_t - x_t q_{t-1}$. Substituting for y_t in the objective function, the problem becomes

$$\max_{q_1, q_2, \dots} \Pi = \sum_{t=0}^{\infty} \beta^t (P(q_t) - c)(q_t - x_t q_{t-1})$$

Each q_t occurs in two terms of this sum, that is

$$\Pi = \cdots + \beta^t (P(q_t) - c)(q_t - x_t q_{t-1}) + \beta^{t+1} (P(q_{t+1}) - c)(q_{t+1} - x_{t+1} q_t)$$

Recalling that $x_t = x\big(P(q_t)\big)$, the first order conditions for an optimal policy are

$$D_{q_t} \Pi = \beta^t \big((p_t - c)(1 - P'x'q_{t-1}) + P'(q_t - x q_{t-1})\big) - \beta^{t+1}(p - c)x = 0$$

In a steady state equilibrium, $q_t = q, p_t = p, x_t = x$ for all t. Dividing by β^t, the equilibrium condition becomes

$$(p - c)(1 - P'x'q) + P'(q - xq) - \beta(p - c)x = 0$$

Rearranging

$$(p - c)(1 - \beta x - x'P'q) = -(1 - x)P'q \tag{7.65}$$

(c) Since $P' < 0$ and $x < 1$ the right hand side of (7.65) is positive. Since $x' > 0$, $x'P'q < 0$ and therefore $1 - \beta x - x'P'q > 1 - \beta x > 0$. Therefore $p > c$.

(d) Dividing (7.65) by p and rearranging

$$\begin{aligned} \frac{p - c}{p} &= -\frac{P'q}{p} \left(\frac{1 - x}{1 - \beta x - x'P'q} \right) \\ &= -\frac{1}{\epsilon} \left(\frac{1 - x}{1 - \beta x - x'P'q} \right) \end{aligned}$$

where

$$\epsilon = \frac{P}{P'q}$$

is the price elasticity of demand. Since $\beta < 1$, $x < 1$, $x' > 0$, $P' < 0$ and $(p - c) > 0$

$$\frac{p - c}{p} < -\frac{1}{\epsilon}$$

which is the optimal markup of monopolist in the absence of recycling. We conclude that recycling lowers the market price and increases the quantity sold.

7.12 The current value Hamiltonian is

$$H(\mathbf{a}(t), s(t), \lambda(t), t) = e^{-rt} f(\mathbf{a}(t), s(t), t) + \lambda(t) g(\mathbf{a}(t), s(t), t)$$

while the initial value Hamiltonian is

$$\tilde{H}(\mathbf{a}(t), s(t), \mu(t), t) = e^{-rt} f(\mathbf{a}(t), s(t), t) + \mu(t) g(\mathbf{a}(t), s(t), t)$$

Letting $\mu(t) = e^{-rt}\lambda(t)$, we can see that the current and initial value Hamiltonians are related by the equations

$$\tilde{H} = e^{-rt} H \text{ and } H = e^{rt} \tilde{H}$$

so that

$$D_s H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) = e^{rt} D_s \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) \tag{7.66}$$

$$D_\lambda H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) = D_\mu \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) = g(\mathbf{a}(t), s(t), t) \tag{7.67}$$

In terms of the current value Hamiltonian, the necessary conditions for optimality are

$$\mathbf{a}^*(t) \text{ maximizes } H\big(\mathbf{a}(t), s(t), \lambda(t), t\big)$$

$$\dot{s} = D_\lambda H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) = g(\mathbf{a}(t), s(t), t) \tag{7.68}$$

$$\dot{\lambda} - r\lambda(t) = -D_s H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) \tag{7.69}$$

$$v'(s(T)) = \lambda(T)$$

Since e^{rt} is monotonic, $\mathbf{a}^*(t)$ maximizes $H\big(\mathbf{a}(t), s(t), \lambda(t), t\big)$ if and only if it maximizes $\tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big)$. Using (7.68) and (7.67)

$$\dot{s} = D_\lambda H\big(\mathbf{a}(t), s(t), \lambda, t\big), t) = D_\mu \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) = g(\mathbf{a}(t), s(t), t)$$

Differentiating $\lambda(t) = e^{rt}\mu(t)$ gives

$$\dot{\lambda} = e^{rt}\dot{\mu} + re^{rt}\mu(t) = e^{rt}\dot{\mu} + r\lambda(t)$$

so that

$$\dot{\lambda} - r\lambda(t) = e^{rt}\dot{\mu}$$

Substituting in (7.69) and using (7.66)

$$e^{rt}\dot{\mu} = \dot{\lambda} - r\lambda(t) = D_s H\big(\mathbf{a}(t), s(t), \lambda(t), t\big) = -e^{rt} D_s \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big)$$

so that

$$\dot{\mu} = D_s \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big)$$

Finally,

$$v'(s(T)) = \lambda(T) = e^{rt}\mu(T)$$

so that

$$e^{-rt} v'(s(T)) = \mu(T)$$

Therefore, we have shown that the necessary conditions for optimality expressed in terms of the initial value Hamiltonian are

$$\mathbf{a}^*(t) \text{ maximizes } \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big)$$

$$\dot{s} = D_\mu \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big) = g(\mathbf{a}(t), s(t), t)$$

$$\dot{\mu} = -D_s \tilde{H}\big(\mathbf{a}(t), s(t), \mu(t), t\big)$$

$$e^{-rt} v'(s(T)) = \mu(T)$$

7.13 The Hamiltonian is

$$H = e^{-rt}\big(p(t)f(k(t)) - qI(t)\big) + \lambda(t)\big(I(t) - \delta k(t)\big)$$

The first-order conditions are

$$H_I = -e^{-rt}q + \lambda(t) = 0 \tag{7.70}$$

$$\dot{k} = I(t) - \delta k(t)$$

$$\dot{\lambda} = -H_k = -e^{-rt}p(t)f'(k(t)) + \delta\lambda(t) \tag{7.71}$$

Equation (7.70) implies

$$\lambda(t) = e^{-rt}q$$

Differentiating

$$\dot{\lambda} = -re^{-rt}q$$

Substituting into (7.71) and using (7.70) yields

$$\begin{aligned} -re^{-rt}q &= -e^{-rt}p(t)f'(k(t)) + \delta\lambda(t) \\ &= -e^{-rt}p(t)f'(k(t)) + \delta e^{-rt}q \end{aligned}$$

Cancelling the common terms and rearranging, we derive the optimality condition

$$p(t)f'(k(t)) = (r + \delta)q$$

7.14

7.15 Bellman's equation is

$$\begin{aligned} v_t(w_t) &= \max_{c_t}\{u(c_t) + \beta v_{t+1}(w_{t+1})\} \\ &= \max_{c_t}\{u(c_t) + \beta v_{t+1}\big((1+r)(w_t - c_t)\big)\} \end{aligned}$$

The first-order condition is

$$u'(c_t) - \beta(1+r)v'_{t+1}\big((1+r)(w_t - c_t)\big) = 0 \tag{7.72}$$

But

$$v_{t+1}(w_{t+1}) = \max_{c_{t+1}}\{u(c_{t+1}) + \beta v_{t+2}((1+r)(w_{t+1} - c_{t+1}))\} \tag{7.73}$$

By the envelope theorem (theorem 6.2)

$$v'_{t+1}(w_{t+1}) = \beta(1+r)v'_{t+2}\big((1+r)(w_{t+1} - c_{t+1})\big) \tag{7.74}$$

The first-order condition for (7.73) is

$$u'(c_{t+1}) - \beta(1+r)v'_{t+2}\big((1+r)(w_{t+1} - c_{t+1})\big) = 0$$

Substituting in (7.74)

$$v'_{t+1}(w_{t+1}) = u'(c_{t+1})$$

and therefore, from (7.72), the optimal policy is characterised by

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$