Inequality constraints

1 Introduction

$$\max_{\mathbf{x}} f(\mathbf{x})$$
$$\text{subject to } g_j(\mathbf{x}) \leq 0, \quad j = 1, 2, \ldots, m$$

Superficially, the extension to inequality constraints is straightforward. However, subtle difficulties arise:

- boundary solutions are more likely and
- regularity is too stringent.

2 Necessary conditions

Let $B(\mathbf{x}^*) = \{j : g_j(\mathbf{x}^*) = 0\}$ denote the set of binding constraints at $\mathbf{x}^*$. If $\mathbf{x}^*$ solves

$$\max f(\mathbf{x}) \text{ subject to } g_j(\mathbf{x}) \leq 0, \quad j = 1, 2, \ldots, m$$

then *a fortiori* it solves

$$\max f(\mathbf{x}) \text{ subject to } g_j(\mathbf{x}) = 0, \quad j \in B(\mathbf{x}^*)$$

By the Lagrange multiplier theorem (Theorem 5.2) there exist multipliers λ_j, $j \in B(\mathbf{x}^*)$ such that

$$\nabla f(\mathbf{x}^*) = \sum_{j \in B(\mathbf{x}^*)} \lambda_j \nabla g_j(\mathbf{x}^*)$$

Further, Proposition 5.2 implies that $\lambda_j \geq 0$. Letting $\lambda_j = 0$ for every $j \notin B(\mathbf{x}^*)$

$$\nabla f(\mathbf{x}^*) = \sum_{j=1}^{m} \lambda_j \nabla g_j(\mathbf{x}^*)$$

with $\lambda_j \geq 0$ **and** $\lambda = 0$ if $g_j(\mathbf{x}^*) < 0$. This can be expressed compactly as

$$\lambda_j g_j(\mathbf{x}^*) = 0 \text{ for every } j$$

which is known as **complementary slackness**.

2.1 Kuhn-Tucker theorem 5.3

Suppose that $\mathbf{x}^*$ is a local optimum of

$$\max_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}$$

and the binding constraints are regular at $\mathbf{x}^*$. Then there exist unique multipliers $\lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ such that

$$\nabla f(\mathbf{x}^*) = \sum_{j=1}^{m} \lambda_j \nabla g_j(\mathbf{x}^*) \text{ and } \lambda_j g_j(\mathbf{x}^*) = 0, \quad j = 1, 2 \dots, m$$

- Alternative proofs
  - perturbation approach (Farkas lemma)
  - slack variables (Exercise 5.32)
- Mixed problems

2.2 Kuhn-Tucker conditions (Corollary 5.3.1)

In application, it is usually more convenient to express these conditions in terms of stationarity of the Lagrangean. Suppose that $\mathbf{x}^*$ is a local solution of

$$\max_{\mathbf{x}} f(\mathbf{x}) \text{ subject to } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}$$

and the binding constraints are regular at $\mathbf{x}^*$. Then there exist unique multipliers $\boldsymbol{\lambda} = \lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ such that the Lagrangean

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x})$$

is stationary at $(\mathbf{x}^*, \lambda)$, that is

$$D_{\mathbf{x}} L[\mathbf{x}^*, \boldsymbol{\lambda}] = D_{\mathbf{x}} f[\mathbf{x}^*] - \sum_{j=1}^{m} \lambda_j D_{\mathbf{x}} g_j[\mathbf{x}^*] = \mathbf{0} \text{ and } \lambda_j g_j(\mathbf{x}^*) = 0 \qquad (1)$$

for every j = 1, 2,. . . , m.

2.3 Nonnegative variables

The optimization problem

$$\max_{\mathbf{x} \geq 0} f(\mathbf{x})$$
$$\text{subject to } \mathbf{g}(\mathbf{x}) \leq \mathbf{0}$$

can be written in standard form as

$$\max_{\mathbf{x}} f(\mathbf{x}) \tag{2}$$
$$\text{subject to } \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \text{ and } \ \mathbf{h}(\mathbf{x}) = -\mathbf{x} \leq \mathbf{0}$$

Suppose that $\mathbf{x}^*$ is a local optimum of (2) and the binding constraints (including the nonnegativity constraint $\mathbf{h}$) are regular at $\mathbf{x}^*$. By the Kuhn-Tucker theorem (Theorem 5.3), there exist $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m) \geq \mathbf{0}$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n) \geq \mathbf{0}$ such that

$$\nabla f(\mathbf{x}^*) = \sum_{j=1}^{m} \lambda_j \nabla g_j(\mathbf{x}^*) + \boldsymbol{\mu} \tag{3}$$

and

$$\lambda_j g_j(\mathbf{x}^*) = 0 \text{ and } \ \mu_i h_i(\mathbf{x}^*) = -\mu_i x_i^* = 0 \tag{4}$$

for every $i = 1, 2, \dots n$ and $j = 1, 2, \dots m$. Since $\boldsymbol{\mu} \geq \mathbf{0}$, (3) implies

$$\nabla f(\mathbf{x}^*) - \sum \lambda_j \nabla g_j(\mathbf{x}^*) \leq \mathbf{0} \tag{5}$$

Further, equation (3) can be solved for $\boldsymbol{\mu}$

$$-\boldsymbol{\mu} = \nabla f(\mathbf{x}^*) - \sum \lambda_j \nabla g_j(\mathbf{x}^*)$$

and substituted into (4)

$$\lambda_j g_j(\mathbf{x}^*) = 0 \text{ and } \ \Big(\nabla f(\mathbf{x}^*) - \sum \lambda_j \nabla g_j(\mathbf{x}^*)\Big)^T \mathbf{x}^* = 0 \tag{6}$$

which should be compared with Corollary 5.2.1.
Now define

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{j=1}^{n} \lambda_j g_j(\mathbf{x})$$

which is the Lagrangean for the problem **ignoring** the nonnegativity constraints. Then (5) can be written as

$$D_{x_i} L[\mathbf{x}^*, \boldsymbol{\lambda}] \leq 0 \text{ for every } \ i = 1, 2, \dots, n$$

and the necessary conditions can be written compactly in a form which emphasizes the symmetry between decision variables $\mathbf{x}$ and Lagrange multipliers $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$.

$$\begin{array}{rrrr} D_{x_i} L[\mathbf{x}^*, \boldsymbol{\lambda}] \leq 0 & x_i^* \geq 0 & x_i^* D_{x_i} L[\mathbf{x}^*, \boldsymbol{\lambda}] = 0 & i = 1, 2, \ldots, n \\ g_j(\mathbf{x}^*) \leq 0 & \lambda_j \geq 0 & \lambda_j g_j(\mathbf{x}^*) = 0 & j = 1, 2, \ldots, m \end{array}$$

Optimality requires that

- for every decision variable x_i, either the Lagrangean is stationary (with respect to x_i) or $x_i = 0$ or both.
- for every constraint, either the constraint is binding or it shadow price is zero or both.

To summarise, to deal compactly with nonnegativity constraints, we may omit them explicitly from the Lagrangean, and then modify first-order conditions appropriately (from (1) to those immediately above), which are also known as the **Kuhn-Tucker conditions**.

2.4 Solving the Kuhn-Tucker conditions

The Kuhn-Tucker conditions are a system of nonlinear equations and inequalities. Solving this system usually involves some trial and error, which may be guided by economic intuition.

Example 5.33 To solve the problem

$$\max_{x_1 \geq 0, x_2 \geq 0} \log x_1 + \log(x_2 + 5)$$
$$\text{subject to } x_1 + x_2 - 4 \leq 0$$

we form the Langrangean

$$L(x_1, x_2, \lambda) = \log x_1 + \log(x_2 + 5) - \lambda(x_1 + x_2 - 4)$$

and derive the Kuhn-Tucker conditions

$$\begin{aligned} D_{x_1} L &= \frac{1}{x_1} - \lambda \leq 0 & x_1 &\geq 0 & x_1\left(\frac{1}{x_1} - \lambda\right) &= 0 \\ D_{x_2} L &= \frac{1}{x_2 + 5} - \lambda \leq 0 & x_2 &\geq 0 & x_2\left(\frac{1}{x_2 + 5} - \lambda\right) &= 0 \\ x_1 + x_2 &\leq 4 & \lambda &\geq 0 & \lambda(4 - x_1 - x_2) &= 0 \end{aligned}$$

There are three cases to consider.

- $x_1 > 0, x_2 > 0 \implies \frac{1}{x_1} = \lambda = \frac{1}{x_2+5} \implies x_1 = x_2 + 5$ which is inconsistent with the constraints.
- $x_1 > 0, x_2 = 0 \implies \lambda = \frac{1}{x_1} > 0 \implies x_1 = 4$
- $x_1 = 0, x_2 > 0$. $x_1 = 0 \implies \lambda = \infty$ while $x_2 > 0 \implies \frac{1}{x_2+5} > 0 \implies x_2 = 4 \implies \lambda = \frac{1}{9}$

The point $x_1 = 4$, $x_2 = 0$ and $\lambda = \frac{1}{4}$ is the unique solution to the Kuhn-Tucker conditions. We note that the constraints are regular at $(4, 0)$ (Exercise 5.35).

Exercise: (Example 5.30) Solve the problem

$$\max_{x_1, x_2} 6x_1 + 2x_1 x_2 - 2x_1^2 - 2x_2^2$$
$$\begin{aligned} \text{subject to } x_1 + 2x_2 - 2 &\leq 0 \\ -x_1 + x_2^2 - 1 &\leq 0 \end{aligned}$$

We look for stationary values of the Lagrangean

$$L(x_1, x_2, \lambda_1, \lambda_2) = 6x_1 + 2x_1x_2 - 2x_1^2 - 2x_2^2 - \lambda_1(x_1 + 2x_2 - 2) - \lambda_2(-x_1 + x_2^2 - 1)$$

which satisfy the complementary slackness conditions. The first-order conditions for stationarity are

$$D_{x_1}L = 6 + 2x_2 - 4x_1 - \lambda_1 + \lambda_2 = 0 \tag{7}$$
$$D_{x_2}L = 2x_1 - 4x_2 - 2\lambda_1 - 2\lambda_2 x_2 = 0 \tag{8}$$

while the complementary slackness conditions are

$$\begin{array}{lll} x_1 + 2x_2 - 2 \leq 0 & \lambda_1 \geq 0 & \lambda_1(x_1 + 2x_2 - 2) = 0 \\ -x_1 + x_2^2 - 1 \leq 0 & \lambda_2 \geq 0 & \lambda_2(-x_1 + x_2^2 + 1) = 0 \end{array}$$

Again, there are three cases to consider.

Case 1 $\lambda_1 > 0, \lambda_2 > 0 \implies x_1 = 0, x_2 = 1$ or $x_1 = 8, x_2 = -3$. Substituting in (7) and (8) gives $\lambda_1 = 3$, $\lambda_2 = -5 < 0$ and $\lambda_1 = -49 < 0$, $\lambda_2 = -17 < 0$ respectively. Neither can be a local optimum.

Case 2 $\lambda_1 > 0, \lambda_2 = 0$. The first-order conditions and the binding constraint constitute a systems of three equations in three unknowns, namely

$$\begin{aligned} 6 + 2x_2 - 4x_1 - \lambda_1 &= 0 \\ 2x_1 - 4x_2 - 2\lambda_1 &= 0 \\ x_1 + 2x_2 &= 2 \end{aligned}$$

which have a unique solution $x_1 = 10/7, x_2 = 2/7, \lambda_1 = 6/7$, which also satisfies the second inequality

$$-\frac{10}{7} + \left(\frac{2}{7}\right)^2 = -\frac{66}{49} < 1$$

This point satisfies the necessary conditions for a local optimum.

Case 3 $\lambda_1 = 0, \lambda_2 > 0$. This yields the system

$$\begin{aligned} 6 + 2x_2 - 4x_1 + \lambda_2 &= 0 \\ 2x_1 - 4x_2 - 2\lambda_2 x_2 &= 0 \\ -x_1 + x_2^2 &= 1 \end{aligned}$$

which has three solutions, but each solution has $\lambda_2 < 0$. Therefore, there cannot be a solution with the first constraint slack.

We conclude that $x_1 = 10/7, x_2 = 2/7$ is the only possible solution of the problem. In fact, this is the unique global solution.

Example 5.32 (Rate of return regulation) The optimization problem of a monopolist subject to rate of return regulation is

$$\begin{aligned} &\max_{k,l} \Pi(k,l) = R(k,l) - rk - wl \\ &\text{subject to } R(k,l) - wl - sk \leq 0 \end{aligned}$$

Assuming the regularity condition is satisfied, optimality requires the Lagrangean

$$L(k,l,\lambda) = R(k,l) - rk - wl - \lambda(R(k,l) - wl - sk)$$

be stationary at the optimal solution, that is

$$\begin{aligned} D_k L &= D_k R(k,l) - r - \lambda D_k R(k,l) + \lambda s = 0 \\ D_l L &= (1-\lambda) D_l R(k,l) - (1-\lambda) w = 0 \end{aligned}$$

with

$$\lambda(R(k,l) - wl - sk) = 0$$

These first-order conditions can be rewritten as

$$\begin{aligned} (1-\lambda) D_k R(k,l) &= (1-\lambda) r - \lambda(s-r) \qquad (9) \\ (1-\lambda) D_l R(k,l) &= (1-\lambda) w \\ \lambda(R(k,l) - wl - sk) &= 0 \end{aligned}$$

Given that $s > r$, (9) ensures that $\lambda \neq 1$. Therefore the first-order conditions can be further simplified to give

$$\begin{aligned} D_k R(k,l) &= r - \frac{\lambda(s-r)}{1-\lambda} \\ D_l R(k,l) &= w \\ \lambda(R(k,l) - wl - sk) &= 0 \end{aligned}$$

In characterizing the optimal solution, we distinguish two cases.

Case 1 $\boxed{\lambda = 0}$ With $\lambda = 0$, the regulatory constraint is not binding and the firm's first-order conditions for profit maximization reduce to those for a unregulated monopolist, namely produce where marginal revenue product is equal to factor cost.

$$\begin{aligned} D_k R(k,l) &= r \\ D_l R(k,l) &= w \end{aligned}$$

Case 2 $\boxed{\lambda > 0}$ In this case, the regulatory constraint is binding. The first-order conditions are

$$D_k R(k,l) = r - \frac{\lambda(s-r)}{1-\lambda}$$
$$D_l R(k,l) = w$$
$$R(k,l) - wl = sk$$

Compared to the unregulated firm, the profit constraint drives a wedge between the marginal revenue product of capital and its price. Using the second-order condition for a maximum, it can be shown that $0 < \lambda < 1$, and therefore that $D_k R(k,l) < r$ at the optimal solution. The firm uses capital inefficiently, in that its marginal revenue product is less than its opportunity cost. It can also be shown (Exercise 5.31) that the regulated monopolist does not produce at minimum cost.

2.5 Upper and lower bounds

A special case which is often encountered is where the constraints take the form of upper and lower bounds, as in the problem

$$\max_{\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}} f(\mathbf{x})$$

which can be written in standard from as

$$\max_{\mathbf{x} \geq \mathbf{0}} f(\mathbf{x}) \text{ subject to } g_i(\mathbf{x}) = x_i - c_i \leq 0, \quad i = 1, 2, \dots, n$$

Assuming $\mathbf{c} > \mathbf{0}$, the binding constraints are regular for all $\mathbf{0} \leq \mathbf{x} \leq \mathbf{c}$. The Kuhn-Tucker conditions are

$$D_{x_i} L[\mathbf{x}^*, \boldsymbol{\lambda}] = D_{x_i} f[\mathbf{x}^*] - \lambda_i \leq 0 \qquad x_i^* \geq 0 \qquad x_i^*\big(D_{x_i} f[\mathbf{x}^*] - \lambda_i\big) = 0$$
$$g_i(\mathbf{x}^*) = x_i^* - c_i \leq 0 \qquad \lambda_i \geq 0 \qquad \lambda_i\big(x_i^* - c_i\big) = 0$$

for every $i = 1, 2, \dots, n$ where L is the Lagrangean $L(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{i=1}^n \lambda_i g_i(\mathbf{x})$. Now for every i

$$x_i^* > 0 \implies D_{x_i} f[\mathbf{x}^*] = \lambda_i \geq 0$$

and

$$x_i^* < c \implies \lambda_i = 0 \implies D_{x_i} f[\mathbf{x}^*] \leq 0$$

These first-order conditions can be expressed very compactly as

$$D_{\mathbf{x}_i} f[\mathbf{x}^*] \geq 0 \text{ if } x_i^* > 0 \text{ and } \; D_{x_i} f[\mathbf{x}^*] \leq 0 \text{ if } x_i^* < c_i$$

for every $i = 1, 2, \dots, n$. These conditions together imply that $D_{\mathbf{x}_i} f[\mathbf{x}^*] = 0$ for every i such that $0 < x_i^* < c_i$.

Example 5.35: Production with capacity constraints

$$\max_{0 \leq y_i \leq Y} \Pi(\mathbf{y}) = R(\mathbf{y}) - c(\mathbf{y})$$

The Kuhn-Tucker conditions for an optimal production plan $\mathbf{y}^*$ are

$$D_{y_i}\Pi[\mathbf{y}^*] \geq 0 \text{ if } y_i^* > 0 \text{ and } \; D_{y_i}\Pi(\mathbf{y}^*) \leq 0 \text{ if } y_i^* < Y$$

for every period i. These require

Off-peak $(y_i^* < Y)$ $R_{y_i} \leq C_{y_i}$ with $R_{y_i} = C_{y_i}$ if $y_i^* > 0$

Peak $(y_i^* = Y)$ $R_{y_i} \geq C_{y_i}$

3 Homework

1. Solve the problem

$$\begin{aligned} \max x_1 x_2 \\ \text{subject to } x_1^2 + 2x_2^2 \leq 3 \\ 2x_1^2 + x_2^2 \leq 3 \end{aligned}$$

2. Derive and interpret the Kuhn-Tucker conditions for the consumer's problem

$$\begin{aligned} \max_{\mathbf{x} \geq \mathbf{0}} u(\mathbf{x}) \\ \text{subject to } \mathbf{p}^T\mathbf{x} \leq m \end{aligned}$$

   constraining consumption to be nonnegative, while allowing the consumer to spend less than her income. We will show later (Example 5.41) that the regularity condition is satisfied.

3. Extend the model of production with capacity constraints above to allow for variable capacity Y and the analyse the necessary conditions for optimality.

Solutions 4

1 The Lagrangean for this problem is

$$L(x_1, x_2, \lambda_1, \lambda_2) = x_1 x_2 - \lambda_1(x_1^2 + 2x_2^2 - 3) - \lambda_2(2x_1^2 + x_2^2 - 3)$$

The first-order conditions for stationarity

$$D_{x_1} L = x_2 - 2\lambda_1 x_1 - 4\lambda_2 x_1 = 0$$
$$D_{x_2} L = x_1 - 4\lambda_1 x_2 - 2\lambda_2 x_2 = 0$$

can be written as

$$x_2 = 2(\lambda_1 + 2\lambda_2)x_1 \quad (1)$$
$$x_1 = 2(2\lambda_1 + \lambda_2)x_2 \quad (2)$$

which must be satisfied along with the complementary slackness conditions

$$x_1^2 + 2x_2^2 - 3 \leq 0 \qquad \lambda_1 \geq 0 \qquad \lambda_1(x_1^2 + 2x_2^2 - 3) = 0$$
$$2x_1^2 + x_2^2 - 3 \leq 0 \qquad \lambda_2 \geq 0 \qquad \lambda_2(2x_1^2 + x_2^2 - 3) = 0$$

First suppose that both constraints are slack so that $\lambda_1 = \lambda_2 = 0$. Then the first-order conditions (1) and (2) imply that $x_1 = x_2 = 0$. $(0, 0)$ satisfies the Kuhn-Tucker conditions. Next suppose that the first constraint is binding while the second constraint is slack ($\lambda_2 = 0$). The first-order conditions (1) and (2) have two solutions, $x_1 = \sqrt{3/2}$, $x_2 = \sqrt{3}/2$, $\lambda = 1/(2\sqrt{2})$ and $x_1 = -\sqrt{3/2}$, $x_2 = -\sqrt{3}/2$, $\lambda = 1/(2\sqrt{2})$, but these violate the second constraint. Similarly, there is no solution in which the first constraint is slack and the second constraint binding. Finally, assume that the both constraints are binding. This implies that $x_1 = x_2 = 1$ or $x_1 = x_2 = -1$, which points satisfy the first-order conditions (1) and (2) with $\lambda_1 = \lambda_2 = 1/6$.
We conclude that three points satisfy the Kuhn-Tucker conditions, namely $(0, 0)$, $(1, 1)$ and $(-1, -1)$. Noting the objective function, we observe that $(0, 0)$ in fact minimizes the objective. We conclude that there are two local maxima, $(1, 1)$ and $(-1, -1)$, both of which achieve the same level of the objective function.

2 The Lagrangean for this problem is

$$L(\mathbf{x}, \lambda) = u(\mathbf{x}) - \lambda(\mathbf{p}^T \mathbf{x} - m)$$

and the first-order (Kuhn-Tucker) conditions are (Corollary 5.3.2)

$$D_{x_i} L[\mathbf{x}^*, \lambda] = D_{x_i} u[\mathbf{x}^*] - \lambda p_i \leq \mathbf{0} \quad \mathbf{x}_i^* \geq 0 \quad x_i^*(D_{x_i} u[\mathbf{x}^*] - \lambda p_i) = 0 \qquad (3)$$

$$\mathbf{p}^T \mathbf{x}^* \leq m \qquad \lambda \geq \mathbf{0} \qquad \lambda(\mathbf{p}^T \mathbf{x}^* - m) = 0 \qquad (4)$$

for every good $i = 1, 2, \ldots, m$. Two cases must be distinguished.

Case 1 $\boxed{\lambda > 0}$ This implies that $\mathbf{p}^T \mathbf{x} = m$, the consumer spends all her income. Condition (3) implies

$$D_{x_i} u[\mathbf{x}^*] \leq \lambda p_i \text{ for every } i \text{ with } D_{x_i} u[\mathbf{x}^*] = \lambda p_i \text{ for every } i \text{ for which } x_i > 0$$

This case was analyzed in Example 5.17.

Case 2 $\boxed{\lambda = 0}$ This allows the possibility that the consumer does not spend all her income. Substituting $\lambda = 0$ in (3) we have $D_{x_i} u[\mathbf{x}^*] = 0$ for every i. At the optimal consumption bundle $\mathbf{x}^*$, the marginal utility of every good is zero. The consumer is satiated, that is no additional consumption can increase satisfaction. This case was analyzed in Example 5.31.

In summary, at the optimal consumption bundle $\mathbf{x}^*$, either

- the consumer is satiated ($D_{x_i} u[\mathbf{x}^*] = 0$ for every i) or
- the consumer consumes only those goods whose marginal utility exceeds the threshold $D_{x_i} u[\mathbf{x}^*] \geq \lambda p_i$ and adjusts consumption so that the marginal utility is proportional to price for all consumed goods.

3 With variable capacity, cost depends upon both output and capacity $c(\mathbf{y}, Y)$ and the monopolist's problem is

$$\max_{\mathbf{y} \geq \mathbf{0}, Y \geq 0} \Pi(\mathbf{y}, Y) = R(\mathbf{y}) - c(\mathbf{y}, Y)$$

subject to the capacity constraints $y_i \leq Y$ for very $i = 1, 2, \dots, n$. Since the upper bound is a decision variable, this problem no longer fits the formulation of Example 5.34 and we apply Corollary 5.3.2. The Lagrangean is

$$L(\mathbf{y}, Y, \boldsymbol{\lambda}) = R(\mathbf{y}) - c(\mathbf{y}, Y) - \sum_{i=1}^{n} \lambda_j (y_i - Y)$$

The Kuhn-Tucker conditions for an optimum require that for every period $i = 1, 2, \dots, n$

$$\begin{aligned} D_{y_i} L = R_{y_i} - c_{y_i} - \lambda_i \leq 0 \qquad & y_i^* \geq 0 \qquad & y_i^*(R_{y_i} - c_{y_i} - \lambda_i) = 0 \\ y_i^* \leq Y \qquad & \lambda_i \geq 0 \qquad & \lambda(y_i^* - Y) = 0 \end{aligned}$$

and that capacity be chosen such that

$$D_Y L = -c_Y + \sum_{i=1}^{n} \lambda_i \leq 0 \qquad Y \geq 0 \qquad Y \left(c_Y - \sum_{i=1}^{n} \lambda_i \right) = 0$$

where $c_Y = D_y c[\mathbf{y}^*, Y]$ is the marginal cost of additional capacity. The conditions for optimal production are the same as before (4), with the enhancement that λ_i measures the degree to which marginal revenue may exceed marginal cost in peak periods. Specifically

$$R_{y_i} = c_{y_i} + \lambda_i$$

The margin λ_i between marginal revenue and marginal cost represents the shadow price of capacity in period i.
The final first-order condition determines the optimal capacity. It says that capacity should be chosen so that the marginal cost of additional capacity is just equal to the total shadow price of capacity in the peak periods

$$c_Y = \sum_{i=1}^{n} \lambda_i$$

An additional unit of capacity would cost c_Y. It would enable the production of an additional unit of electricity in each peak period, the net revenue from which is given by $\lambda_i = R_{y_i} - c_{y_i}$. Total net revenue of an additional unit of capacity would therefore be $\sum_{i=1}^{n} \lambda_i$, which at the optimal capacity, should be precisely equal to its marginal cost.