Concave and convex functions

1 Concave and convex functions

1.1 Definition

▷ $f\colon S \to \Re$ is **concave** if for every $\mathbf{x}_1, \mathbf{x}_2$ in S

$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \geq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \text{ for every } \; 0 \leq \alpha \leq 1$$

It is **strictly concave** if the inequality is strict, and **convex** if the inequality is reversed.

▷ Equivalently, a function $f\colon S \to \Re$ is concave if and only if

- hypo $f = \{(\mathbf{x}, y) \in S \times \Re : y \leq f(\mathbf{x}), \mathbf{x} \in S\}$ is convex (Proposition 3.7, Exercise 3.125)
- $f(\mathbf{x}) \leq f(\mathbf{x}_0) + \nabla f^T(\mathbf{x} - \mathbf{x}_0)$ for every $\mathbf{x}, \mathbf{x}_0 \in S$ (Exercise 4.67)

▷ $f\colon X \to \Re$ is **locally concave** at $\mathbf{x}_0$ if there exists a convex neighborhood S of $\mathbf{x}_0$ such that for every $\mathbf{x}_1, \mathbf{x}_2 \in S$

$$f(\alpha \mathbf{x}_1 + (1-\alpha)\mathbf{x}_2) \geq \alpha f(\mathbf{x}_1) + (1-\alpha) f(\mathbf{x}_2) \text{ for every } \; 0 \leq \alpha \leq 1$$

f is concave if and only if it is locally concave at every $\mathbf{x} \in X$.

1.2 Examples

▷ x^2, e^x are convex.

▷ $-x^2$, $\log x$ are concave.

▷ Cobb-Douglas $f(\mathbf{x}) = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ is concave if $\sum a_i \leq 1$.

▷ CES $f(\mathbf{x}) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho + \ldots \alpha_n x_n^\rho)^{1/\rho}$ is concave if $\rho \leq 1$ and convex otherwise

▷ The profit function $\Pi(\mathbf{p}) = \max_{\mathbf{y} \in Y} \sum_i p_i y_i$ is convex.

▷ The cost function $c(\mathbf{w}, y) = \min_{\mathbf{x} \in V(y)} \sum_i w_i x_i$ is concave in $\mathbf{w}$

1.3 Properties

If f and g are concave then

- ▷ $-f$ is convex (Exercise 3.124)
- ▷ $1/f$ is convex if $f > 0$ (Exercise 3.135)
- ▷ $1/f$ is concave if $f < 0$ (Exercise 3.135)
- ▷ f^{-1} is convex (Example 3.45)
- ▷ $f + g$ is concave (Exercise 3.131)
- ▷ αf is concave for every $\alpha \geq 0$ (Exercise 3.131)
- ▷ $g \circ f$ is concave if g is increasing (Exercise 3.133)
- ▷ $\log f$ concave (Example 3.51)
- ▷ f is continuous on the interior of its domain† (Corollary 3.8.1)
- ▷ f is differentiable almost everywhere† (Remark 4.14)

1.4 Identification

- ▷ Plotting the function (e.g. *Mathematica*)
- ▷ Apply properties to combinations of known functions (Examples 3.73 and 3.74)
- ▷ Hessian matrix (Proposition 4.1)

$$f \text{ is locally} \begin{Bmatrix} \text{convex} \\ \text{concave} \end{Bmatrix} \text{at } \mathbf{x} \iff H_f(\mathbf{x}) \text{ is } \begin{Bmatrix} \text{nonnegative} \\ \text{nonpositive} \end{Bmatrix} \text{definite}$$

$$f \text{ is strictly locally} \begin{Bmatrix} \text{convex} \\ \text{concave} \end{Bmatrix} \text{at } \mathbf{x} \text{ if } H_f(\mathbf{x}) \text{ is } \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \text{definite.}$$

The definiteness of the Hessian can be assessed by

- eigenvalues (Exercise 3.96)
- determinantal tests (Simon & Blume 381-386, Varian 475-477)

2 Quasiconcavity

2.1 Definition

▷ f is **quasiconcave** if

$$f(\alpha\mathbf{x}_1+(1-\alpha)\mathbf{x}_2) \geq \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\} \text{ for every } \mathbf{x}_1, \mathbf{x}_2 \in S \text{ and } 0 \leq \alpha \leq 1$$

f is **quasiconvex** if

$$f(\alpha\mathbf{x}_1+(1-\alpha)\mathbf{x}_2) \leq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\} \text{ for every } \mathbf{x}_1, \mathbf{x}_2 \in S \text{ and } 0 \leq \alpha \leq 1$$

▷ Equivalently, f is quasiconcave if and only if every upper contour set is convex, that is $\succsim_f(c) = \{\, \mathbf{x} \in X : f(\mathbf{x}) \geq c \,\}$ is convex for every $c \in \Re$. Similarly, f is quasiconvex if and only if every lower contour set is convex, that is $\precsim_f(c) = \{\, \mathbf{x} \in X : f(\mathbf{x}) \leq c \,\}$ is convex for every $a \in \Re$.

2.2 Examples

▷ Every concave function is quasiconcave.

▷ x^3 is quasiconcave and quasiconvex.

▷ Cobb-Douglas $f(\mathbf{x}) = x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ is quasiconcave

▷ CES $\left(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho + \dots \alpha_n x_n^\rho\right)^{1/\rho}$ is quasiconcave if $\rho \leq 1$

▷ The utility function $u(\mathbf{x})$ is quasiconcave if preferences are convex.

▷ The indirect utility function $v(\mathbf{p}, m) = \max_{\mathbf{x}} u(\mathbf{x})$ is quasiconvex in $\mathbf{p}$.

2.3 Properties

If f is quasiconcave then

▷ $-f$ is quasiconvex (Exercise 3.143)

▷ $g \circ f$ is quasiconcave if g is increasing (Exercise 3.148)

▷ fg is quasiconcave if g is quasiconcave (Exercise 3.153)

▷ $f + g$ is ????? (Example 3.57)

▷ f is concave if $f > 0$ and homogenous of degree $k \leq 1$ (Proposition 3.12)

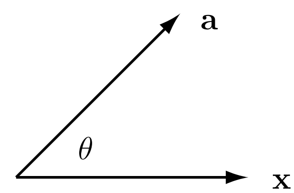
2.4 Identification

- ▷ Plot contours
- ▷ Apply properties to combinations of known functions (Exercises 3.148 and 3.151 to 3.153)
- ▷ Bordered Hessian matrix (Simon & Blume 386-393)

3 Inner product

Given two vectors $\mathbf{a}, \mathbf{x} \in \Re^n$, the angle between them is given by

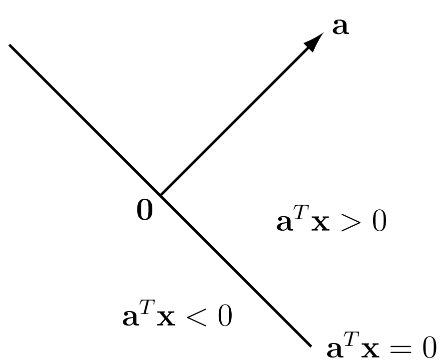
$$\cos\theta = \frac{\mathbf{a}^T\mathbf{x}}{\|\mathbf{a}\|\,\|\mathbf{x}\|}$$



where

$$\mathbf{a}^T\mathbf{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n = \sum_{i=1}^{n} a_ix_i \text{ and } \quad \|\mathbf{x}\| = \sqrt{\mathbf{a}^T\mathbf{x}}$$

The orthogonal hyperplane $\mathbf{a}^T\mathbf{x} = 0$ divides the space into two halfspaces, containing respectively those vectors that make acute and obtuse angles with $\mathbf{a}$.



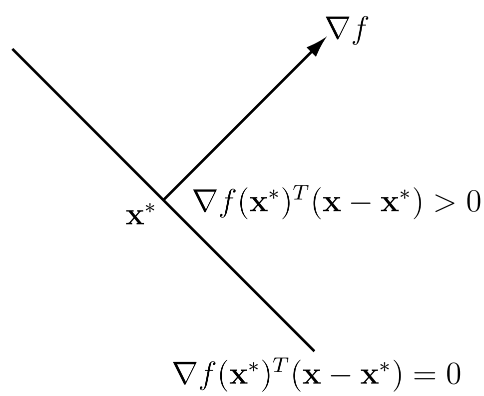
The gradient of a functional $f \colon \Re^n \to \Re$ at $\mathbf{x}^*$

$$\nabla f(\mathbf{x}^*) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

points in the direction of steepest ascent. For any $\mathbf{x}$, the inner product

$$\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}^*)(x_i - x_i^*)$$

measures the angle between $\nabla f(\mathbf{x}^*)$ and $(\mathbf{x} - \mathbf{x}^*)$, and the orthogonal hyperplane divides the space accordingly.



4 Pseudoconcavity

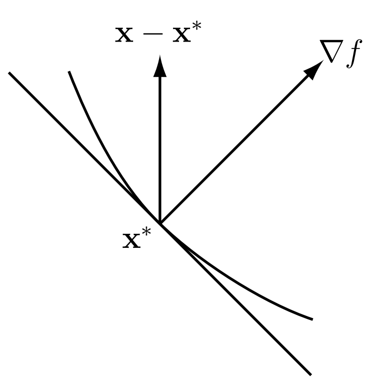
4.1 Definition

▷ f is **quasiconcave** if and only if

$$f(\mathbf{x}) \geq f(\mathbf{x}^*) \implies \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0$$

▷ f is **pseudoconcave** if

$$f(\mathbf{x}) > f(\mathbf{x}^*) \implies \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) > 0$$



4.2 Examples

- ▷ $f(x) = x^3$ is strictly quasiconcave but not pseudoconcave
  - $f(1) > f(0)$ but $\nabla f(0)(1-0) = 0$
- ▷ Cobb-Douglas is pseudoconcave
- ▷ The utility function $u(\mathbf{x})$ is pseudoconcave if preferences are convex **and** nonsatiated

4.3 Properties

- ▷ concave $\implies$ pseudoconcave $\implies$ quasiconcave
- ▷ Every regular quasiconcave function is pseudoconcave
- ▷ A pseudoconcave function is locally concave at every stationary point

5 Linear and quadratic approximation (Taylor's theorem)

For smooth functions

$$f(\mathbf{x}_0 + \mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^T \mathbf{x} \qquad \Big(+ \eta(\mathbf{x}) \left\| \mathbf{x} \right\|, \quad \eta(\mathbf{x}) \to 0 \text{ as } \mathbf{x} \to \mathbf{0} \Big)$$

$$f(\mathbf{x}_0 + \mathbf{x}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^T \mathbf{x} + \tfrac{1}{2}\mathbf{x}^T H_f(\mathbf{x}_0)\mathbf{x} \qquad \Big(+ \eta_2(\mathbf{x}) \left\| \mathbf{x} \right\|^2 \Big)$$

where

- $\nabla f(\mathbf{x}_0)^T \mathbf{x} = \sum_{i=1}^n D_{x_i} f[\mathbf{x}^0] x_i = \sum_{i=1}^n \frac{\partial f[\mathbf{x}^0]}{\partial x_i} x_i$
- $\mathbf{x}^T H_f(\mathbf{x}_0)\mathbf{x} = \sum_i \sum_j D^2_{x_i x_j} f[\mathbf{x}_0] x_i x_j = \sum_i \sum_j \frac{\partial^2 f[\mathbf{x}_0]}{\partial x_i \partial x_j} x_i x_j$

See Section 4.4, especially Theorem 4.3, Corollary 4.3.1. and Example 4.33.

Note: $Df[\mathbf{x}_0]$ denotes the derivative of f at $\mathbf{x}_0$ (a linear function), $Df[\mathbf{x}_0](\mathbf{x})$ denotes its value at $\mathbf{x}$. $D_{x_i} f[\mathbf{x}_0]$ denotes the partial derivative of f with respect to x_i ($= \partial f / \partial x_i$) and $D^2_{x_i x_j} f[\mathbf{x}_0]$ denotes the second partial derivative evaluated at $\mathbf{x}_0$. (See Remark 4.3, p.8 and Remark 4.7. p.15).

6 Quadratic forms

Given a square, symmetrix matrix A, the function

$$q(\mathbf{x}) = \mathbf{x}^T A\, \mathbf{x} = (x_1, x_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

is called a **quadratic form**.

$$q \text{ is } \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \text{ definite if} \begin{Bmatrix} q(\mathbf{x}) > 0 \\ q(\mathbf{x}) < 0 \end{Bmatrix} \text{for every } \mathbf{x} \neq 0$$

Similarly, we say that the matrix A is positive (negative) definitive if its quadratic form is positive (negative) definite.
"Completing the square", q can be rewritten as

$$q(\mathbf{x}) = \mathbf{x}^T A\,\mathbf{x} = a_{11}(x_1 + \frac{a_{12}}{a_{11}} x_2)^2 + (\frac{a_{11}a_{22} - a_{12}^2}{a_{11}})x_2^2$$

From this, we can deduce that

$$q \text{ is } \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \text{ definite if} \begin{Bmatrix} a_{11} > 0 \\ a_{11} < 0 \end{Bmatrix} \text{ and } a_{11}a_{22} > a_{12}^2$$

Similarly

$$q \text{ is } \begin{Bmatrix} \text{nonnegative} \\ \text{nonpositive} \end{Bmatrix} \text{ definite if} \begin{Bmatrix} a_{11}, a_{22} \geq 0 \\ a_{11}, a_{22} \leq 0 \end{Bmatrix} \text{ and } a_{11}a_{22} \geq a_{12}^2$$

The Hessian H of a C^2 function f is a square, symmetric matrix

$$H(\mathbf{x}) = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$$

H is nonpositive definitive if and only if

$$f_{11} \leq 0, \quad f_{22} \leq 0, \text{ and } f_{11}f_{22} \geq f_{12}^2$$

For the Cobb-Douglas function

$$f = x_1^{a_1} x_2^{a_2}$$

$$\begin{aligned} f_{11} &= a_1(a_1 - 1)x_1^{a_1-2}x_2^{a_2} \\ f_{22} &= a_2(a_2 - 1)x_1^{a_1}x_2^{a_2-2} \\ f_{12} &= a_1 a_2 x_1^{a_1-1}x_2^{a_2-1} \end{aligned}$$

Given $x_1, x_2 \geq 0$

$$\begin{aligned} f_{11} \leq 0 &\quad \text{if and only if } 0 \leq a_1 \leq 1 \\ f_{22} \leq 0 &\quad \text{if and only if } 0 \leq a_2 \leq 1 \end{aligned}$$

Under these preceding conditions

$$\begin{aligned} f_{11}f_{22} - f_{12}^2 &= (a_1a_2(a_1 - 1)(a_2 - 1) - (a_1a_2)^2)x_1^{2a_1-2}x_2^{2a_2-2} \\ &= (a_1a_2(a_1a_2 - a_1 - a_2 + 1 - a_1a_2))x_1^{2a_1-2}x_2^{2a_2-2} \\ &= a_1a_2(1 - a_1 - a_2)x_1^{2a_1-2}x_2^{2a_2-2} \\ &\geq 0 \text{ provided } a_1 + a_2 \leq 1 \end{aligned}$$

7 Homework

1. Show that the cost function $c(\mathbf{w}, y)$ of a competitive firm (Example 2.31) is concave in input prices $\mathbf{w}$.

2. Show that the indirect utility function is quasiconvex in $\mathbf{p}$. [Hint: Show that the lower contour sets $\precsim_v(c) = \{\, \mathbf{p} : v(\mathbf{p}, m) \leq c \}$ are convex for every c.]

3. Is the CES function $f(\mathbf{x}) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho + \dots \alpha_n x_n^\rho)^{1/\rho}$ pseudoconcave?

4. Show

   (a) Every differentiable concave function is pseudoconcave.

   (b) Every pseudoconcave function is quasiconcave

   (c) Every regular quasiconcave function is pseudoconcave.

Solutions: Concave and convex functions

1 Suppose that $\mathbf{x}^1$ minimizes the cost of producing y at input prices $\mathbf{w}^1$ while $\mathbf{x}^2$ minimizes cost at $\mathbf{w}^2$. For some $\alpha \in [0,1]$, let $\bar{\mathbf{w}}$ be the weighted average price, that is

$$\bar{\mathbf{w}} = \alpha \mathbf{w}^1 + (1-\alpha)\mathbf{w}^2$$

and suppose that $\bar{\mathbf{x}}$ minimizes cost at $\bar{\mathbf{w}}$. Then

$$\begin{aligned} c(\bar{\mathbf{w}}, y) &= \bar{\mathbf{w}}\bar{\mathbf{x}} \\ &= (\alpha \mathbf{w}^1 + (1-\alpha)\mathbf{w}^2)\bar{\mathbf{x}} \\ &= \alpha \mathbf{w}^1\bar{\mathbf{x}} + (1-\alpha)\mathbf{w}^2\bar{\mathbf{x}} \end{aligned}$$

But since $\mathbf{x}^1$ and $\mathbf{x}^2$ minimize cost at $\mathbf{w}^1$ and $\mathbf{w}^2$ respectively

$$\begin{aligned} \alpha \mathbf{w}^1\bar{\mathbf{x}} \geq \alpha \mathbf{w}^1\mathbf{x}^1 = \alpha c(\mathbf{w}^1, y) \\ (1-\alpha)\mathbf{w}^2\bar{\mathbf{x}} \geq (1-\alpha)\mathbf{w}^2\mathbf{x}^2 = (1-\alpha)c(\mathbf{w}^2, y) \end{aligned}$$

so that

$$c(\bar{\mathbf{w}}, y) = c(\alpha\mathbf{w}^1 + (1-\alpha)\mathbf{w}^2, y) = \alpha\mathbf{w}^1\bar{\mathbf{x}} + (1-\alpha)\mathbf{w}^2\bar{\mathbf{x}} \geq \alpha c(\mathbf{w}^1, y) + (1-\alpha)c\mathbf{w}^2, y)$$

This establishes that the cost function c is concave in $\mathbf{w}$.

2 For given c and m, choose any $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\precsim_v(c)$. For any $0 \leq \alpha \leq 1$, let $\bar{\mathbf{p}} = \alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_2$. The key step is to show that any commodity bundle $\mathbf{x}$ which is affordable at $\bar{\mathbf{p}}$ is also affordable at either $\mathbf{p}_1$ or $\mathbf{p}_2$. Assume that $\mathbf{x}$ is affordable at $\bar{\mathbf{p}}$, that is $\mathbf{x}$ is in the budget set

$$\mathbf{x} \in X(\bar{\mathbf{p}}, m) = \{\, \mathbf{x} : \bar{\mathbf{p}}\mathbf{x} \leq m \,\}$$

To show that $\mathbf{x}$ is affordable at either $\mathbf{p}_1$ or $\mathbf{p}_2$, that is

$$\mathbf{x} \in X(\mathbf{p}_1, m) \text{ or } \mathbf{x} \in X(\mathbf{p}_2, m)$$

assume to the contrary that

$$\mathbf{x} \notin X(\mathbf{p}_1, m) \text{ and } \ \mathbf{x} \notin X(\mathbf{p}_2, m)$$

This implies that

$$\mathbf{p}_1\mathbf{x} > m \text{ and } \ \mathbf{p}_2\mathbf{x} > m$$

so that

$$\alpha\mathbf{p}_1\mathbf{x} > \alpha m \text{ and } \ (1-\alpha)\mathbf{p}_2 > (1-\alpha)m$$

Summing these two inequalities

$$\bar{\mathbf{p}}\mathbf{x} = (\alpha\mathbf{p}_1 + (1-\alpha)\mathbf{p}_2)\mathbf{x} > m$$

contradicting the assumption that $\mathbf{x} \in X(\bar{\mathbf{p}}, m)$. We conclude that

$$X(\bar{\mathbf{p}}, m) \subseteq X(\mathbf{p}_1, m) \cup X(\mathbf{p}_2, m)$$

Now

$$\begin{aligned} v(\bar{p}, m) &= \sup\{\, u(\mathbf{x}) : \mathbf{x} \in X(\bar{\mathbf{p}}, m)\,\} \\ &\leq \sup\{\, u(\mathbf{x}) : \mathbf{x} \in X(\mathbf{p}_1, m) \cup X(\mathbf{p}_2, m)\,\} \\ &\leq c \end{aligned}$$

Therefore $\bar{\mathbf{p}} \in \precsim_v(c)$ for every $0 \leq \alpha \leq 1$. Thus, $\precsim_v(c)$ is convex and so v is quasiconvex (Exercise 3.146).

3 The CES function is quasiconcave provided $\rho \leq 1$ (Exercise 3.58). Since $D_{x_i} f(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Re^n_+ +$, the CES function with $\rho \leq 1$ is pseudoconcave on $\Re^n_{++}$.

4 (a) If $f \in F[S]$ is concave (and differentiable)

$$f(\mathbf{x}) \leq f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0)$$

for every $\mathbf{x}, \mathbf{x}_0 \in S$. Therefore

$$f(\mathbf{x}) > f(\mathbf{x}_0) \implies \nabla f(\mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) > 0$$

f is pseudoconcave.

(b) Assume to the contrary that f is pseudoconcave but not quasiconcave. Then, there exists points $\mathbf{x}_1$, $\mathbf{x}_2$ and $\bar{\mathbf{x}} = \alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$, $\mathbf{x}_1, \mathbf{x}_2 \in S$ such that

$$f(\bar{\mathbf{x}}) < \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\} \tag{1}$$

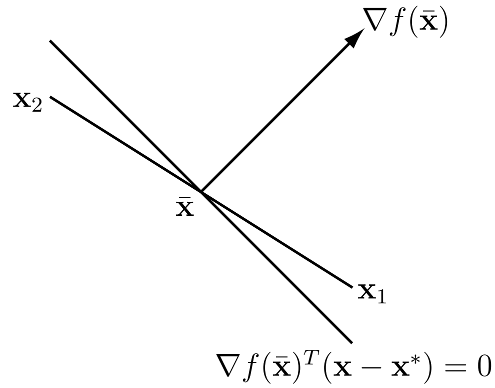
Pseudoconcavity implies

$$\nabla f(\bar{\mathbf{x}})^T(\mathbf{x}_1 - \bar{\mathbf{x}}) > 0 \tag{2}$$

that is $\mathbf{x}_1$ lies on the positive side of the orthogonal hyperplane $\nabla f(\bar{\mathbf{x}})^T(\mathbf{x} - \bar{\mathbf{x}}) = 0$. Therefore, $\mathbf{x}_2$ must lie on the negative side, that is

$$\nabla f(\bar{\mathbf{x}})^T(\mathbf{x}_2 - \bar{\mathbf{x}}) < 0$$

contradicting the pseudconcavity of f.



More precisely, since $\alpha \mathbf{x}_1 = \bar{\mathbf{x}} - (1-\alpha)\mathbf{x}_2$

$$\alpha(\mathbf{x}_1 - \bar{\mathbf{x}}) = (1-\alpha)(\bar{\mathbf{x}} - \mathbf{x}_2)$$

and therefore

$$\mathbf{x}_1 - \bar{\mathbf{x}} = -\frac{1-\alpha}{\alpha}(\mathbf{x}_2 - \bar{\mathbf{x}})$$

Substituting in (2) gives

$$\frac{1-\alpha}{\alpha} \nabla f(\bar{\mathbf{x}})^T(\mathbf{x}_2 - \bar{\mathbf{x}}) < 0$$

which by pseudoconcavity implies $f(\mathbf{x}_2) \leq f(\bar{\mathbf{x}})$ contradicting our assumption (1) .

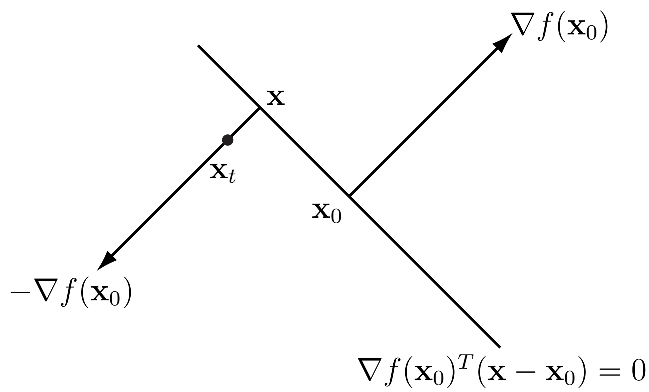
(c) Suppose to the contrary that f is regular and quasiconcave but not pseudoconcave, so that there exists $\mathbf{x}, \mathbf{x}_0$ such that

$$f(\mathbf{x}) > f(\mathbf{x}_0) \text{ and } \nabla f(\mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) \leq 0$$

Since f is regular, $\nabla f(\mathbf{x}_0) \neq 0$ and we can move a small distance from $\mathbf{x}$ in the direction opposite to $\nabla f(\mathbf{x}_0)$ to find a point $\mathbf{x}'$ at which

$$f(\mathbf{x}') > f(\mathbf{x}_0) \text{ and } \nabla f(\mathbf{x}_0)^T(\mathbf{x}' - \mathbf{x}_0) < 0$$

contradicting the assumed quasiconcavity of f.



To make this precise, for every $t \in \Re_+$, let

$$\mathbf{x}_t = \mathbf{x} - t\nabla f(\mathbf{x}_0)$$

Then

$$\begin{aligned}\nabla f(\mathbf{x}_0)^T(\mathbf{x}_t - \mathbf{x}_0) &= \nabla f(\mathbf{x}_0)^T\big(\mathbf{x} - t\nabla f(\mathbf{x}_0) - \mathbf{x}_0\big)\\ &= \nabla f(\mathbf{x}_0)^T(\mathbf{x} - \mathbf{x}_0) - t\nabla f(\mathbf{x}_0)^T\nabla f(\mathbf{x}_0)\\ &\leq -t\,\|\nabla f(\mathbf{x}_0)\|^2 < 0\end{aligned}$$

for every $t \in \Re_+$ since f is regular. Since $f(\mathbf{x}) > f(\mathbf{x}_0)$ and f is continuous, there exists $t > 0$ such that

$$f(\mathbf{x}_t) > f(\mathbf{x}_0) \text{ and } \nabla f(\mathbf{x}_0)^T(\mathbf{x}_t - \mathbf{x}_0) < 0$$

contradicting the quasiconcavity of f.