Comparative statics

1 The maximum theorems

$$\max_{\mathbf{x} \in G(\boldsymbol{\theta})} f(\mathbf{x}, \boldsymbol{\theta})$$

Let

$$v(\boldsymbol{\theta}) = \max_{\mathbf{x} \in G(\boldsymbol{\theta})} f(\mathbf{x}, \boldsymbol{\theta}) \qquad \varphi(\boldsymbol{\theta}) = \arg \max_{\mathbf{x} \in G(\boldsymbol{\theta})} f(\mathbf{x}, \boldsymbol{\theta})$$

	Monotone maximum theorem **Theorem 2.1**	Continuous maximum theorem **Theorem 2.3**	Convex maximum theorem **Theorem 3.10**	Smooth maximum theorem **Theorem 6.1**
Objective function	supermodular, increasing	continuous	concave	smooth
Constraint correspondence	weakly increasing	continuous, compact-valued	convex	smooth regular
Value function	increasing	continuous	concave	locally smooth
Solution correspondence	increasing	compact-valued nonempty, uhc	convex-valued	locally smooth

2 The envelope theorems

2.1 Envelope theorem 1

$$\begin{aligned} v(\boldsymbol{\theta}) &= \max_{x^* \in G} f(\mathbf{x}, \boldsymbol{\theta}) \\ &= f(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) \end{aligned}$$

so that

$$v'(\boldsymbol{\theta}) = f_{\mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \boldsymbol{\theta}} + f_{\boldsymbol{\theta}}$$

The first-order conditions determining $\mathbf{x}^*$ are

$$f_{\mathbf{x}} = \lambda g_{\mathbf{x}}$$

Moveover, $\mathbf{x}^*(\boldsymbol{\theta})$ satisfies the constraint as a identity

$$g(\mathbf{x}^*(\boldsymbol{\theta})) = 0 \implies g_{\mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \boldsymbol{\theta}} = 0$$

Substituting, we conclude that

$$v'(\boldsymbol{\theta}) = f_{\boldsymbol{\theta}}$$

Example 1 (Chip producer) It is characteristic of microchip production technology that a proportion of output is defective. Consider a small producer for whom the price of good chips p is fixed. Suppose that proportion $1 - \theta$ of the firm's chips are defective and cannot be sold. Let $c(y)$ denote the firm's total cost function where y is the number of chips (including defectives) produced. Suppose that with experience, the yield of good chips θ increases. How does this affect the firm's production y? Does the firm compensate for the increased yield by reducing production, or does it celebrate by increasing production?
The firm's optimization problem is

$$\begin{aligned} v(\theta) &= \max_y \theta p y - c(y) \\ &= \theta p y^* - c(y^*) \\ v'(\theta) &= p y^* + \theta p \frac{\partial y^*}{\partial \theta} - c'(y^*) \frac{\partial y^*}{\partial \theta} \\ &= p y^* + (\theta p - c'(y^*)) \frac{\partial y^*}{\partial \theta} \end{aligned}$$

But the first-order condition defining $y^*(\theta)$ is

$$\theta p - c'(y^*) = 0$$

so that

$$v'(\theta) = p y^* > 0$$

Further, we can deduce that

$$y^*(\theta) = \frac{v'(\theta)}{p}$$

so that

$$\frac{\partial y^*(\theta)}{\partial \theta} = \frac{v''(\theta)}{p} \geq 0$$

since the profit function is convex.

2.2 Envelope theorem 2

$$\begin{aligned} v(\boldsymbol{\theta}) &= \max_{x^* \in G(\boldsymbol{\theta})} f(\mathbf{x}, \boldsymbol{\theta}) \\ &= f(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) \\ v'(\boldsymbol{\theta}) &= f_{\mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \boldsymbol{\theta}} + f_{\boldsymbol{\theta}} \end{aligned}$$

The first-order conditions determining $\mathbf{x}^*$ are

$$f_{\mathbf{x}} = \lambda g_{\mathbf{x}}$$

Moveover, $\mathbf{x}^*(\boldsymbol{\theta})$ satisfies the constraint as a identity

$$g(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) = 0 \implies g_{\mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \boldsymbol{\theta}} + g_{\boldsymbol{\theta}} = 0$$

or

$$g_{\mathbf{x}} \frac{\partial \mathbf{x}^*}{\partial \boldsymbol{\theta}} = -g_{\boldsymbol{\theta}}$$

Substituting, we conclude that

$$v'(\boldsymbol{\theta}) = f_{\boldsymbol{\theta}} - \lambda g_{\boldsymbol{\theta}} = L_{\boldsymbol{\theta}}$$

Example 2 (Consumer problem)

$$v(\mathbf{p}, m) = \max_{\mathbf{x} \in X} u(\mathbf{x}) \tag{1}$$

$$\text{subject to } \mathbf{p}^T \mathbf{x} = m \tag{2}$$

$$L = u(\mathbf{x}) - \lambda(\mathbf{p}^T \mathbf{x} - m)$$

$$\begin{aligned} \frac{\partial v}{\partial m} &= L_m = \lambda \\ \frac{\partial v}{\partial p_i} &= L_{p_i} = -\lambda x_i^* \end{aligned}$$

which leads immediately to Roy's identity

$$\mathbf{x}_i^*(\mathbf{p}, m) = -\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial m}}$$

2.3 Smooth envelope theorem (Corollary 6.1.1)

Assume that $\mathbf{x}_0$ is a strict local maximum of

$$\max_{\mathbf{x} \in G(\boldsymbol{\theta})} f(\mathbf{x}, \boldsymbol{\theta})$$

where $G(\boldsymbol{\theta}) = \{\, \mathbf{x} \in X : \mathbf{g}(\mathbf{x}, \boldsymbol{\theta}) \leq 0 \,\}$. By the smooth maximum theorem, there exists a neighbourhood Ω around $\boldsymbol{\theta}_0$ and function $\mathbf{x}^*$ such that

$$v(\boldsymbol{\theta}) = f(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) \text{ for every } \boldsymbol{\theta} \in \Omega$$

and v is differentiable. Applying the chain rule

$$D_{\boldsymbol{\theta}} v[\boldsymbol{\theta}] = \underset{\substack{\uparrow \\ \text{indirect}}}{f_{\mathbf{x}} \mathbf{x}^*_{\boldsymbol{\theta}}} + \underset{\substack{\uparrow \\ \text{direct}}}{f_{\boldsymbol{\theta}}}$$

What do we know of the indirect effect?

First If $\mathbf{x}^*$ is optimal, it must satisfy the Kuhn-Tucker conditions

$$f_{\mathbf{x}} = \boldsymbol{\lambda}_0^T \mathbf{g}_{\mathbf{x}} \text{ and } \boldsymbol{\lambda}_0^T \mathbf{g}(\mathbf{x}, \boldsymbol{\theta}) = 0 \tag{3}$$

at $(\mathbf{x}_0, \boldsymbol{\lambda}_0)$ where $\boldsymbol{\lambda}_0$ is the unique Lagrange multiplier associated with $\mathbf{x}_0$.

Second The solution $\mathbf{x}^*(\boldsymbol{\theta})$ satisfies the constraint $\mathbf{g}(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) = \mathbf{0}$ for all $\boldsymbol{\theta} \in \Omega$. Another application of the chain rule gives

$$\mathbf{g}_{\mathbf{x}} \mathbf{x}^*_{\boldsymbol{\theta}} + \mathbf{g}_{\boldsymbol{\theta}} = \mathbf{0} \implies \boldsymbol{\lambda}_0^T \mathbf{g}_{\mathbf{x}} \mathbf{x}^*_{\boldsymbol{\theta}} = -\boldsymbol{\lambda}^T \mathbf{g}_{\boldsymbol{\theta}} \tag{4}$$

Using (3) and (4), the indirect effect is $f_{\mathbf{x}} \mathbf{x}^*_{\boldsymbol{\theta}} = \boldsymbol{\lambda}_0^T g_{\mathbf{x}} \mathbf{x}^*_{\boldsymbol{\theta}} = -\boldsymbol{\lambda}^T \mathbf{g}_{\boldsymbol{\theta}}$ and therefore

$$D_{\boldsymbol{\theta}} v[\boldsymbol{\theta}] = f_{\boldsymbol{\theta}} - \boldsymbol{\lambda}_0^T \mathbf{g}_{\boldsymbol{\theta}} = L_{\boldsymbol{\theta}} \tag{5}$$

where L denotes the Lagrangean $L(\mathbf{x}, \boldsymbol{\theta}, \boldsymbol{\lambda}) = f(\mathbf{x}, \boldsymbol{\theta}) - \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x}, \boldsymbol{\theta})$. This is the **envelope theorem**, which states that the derivative of the value function is equal to the partial derivative of the Lagrangean evaluated at the optimal solution $(\mathbf{x}_0, \boldsymbol{\lambda}_0)$.

In the special case in which the feasible set G is independent of the parameters, $\mathbf{g}_{\boldsymbol{\theta}} = \mathbf{0}$ and (5) becomes

$$D_{\boldsymbol{\theta}} v[\boldsymbol{\theta}] = f_{\boldsymbol{\theta}}$$

The indirect effect is zero, and the only impact on v of a change in $\boldsymbol{\theta}$ is the direct effect $\mathbf{f}_{\boldsymbol{\theta}}$.

2.4 General envelope theorem (Theorem 6.2)

The assumptions required for Corollary 6.1.1 are stringent. Where the feasible set is independent of the parameters, a more general result can be given. Let $\mathbf{x}^*$ be the solution correspondence of the constrained optimization problem

$$\max_{\mathbf{x}\in G} f(\mathbf{x}, \boldsymbol{\theta})$$

in which $f\colon G \times \Theta \to \Re$ is continuous and G compact. Suppose that f is continuously differentiable in θ, that is $D_{\boldsymbol{\theta}} f[\mathbf{x}, \boldsymbol{\theta}]$ is continuous in $G \times \Theta$. Then the value function

$$v(\theta) = \sup_{x\in G} f(\mathbf{x}, \boldsymbol{\theta})$$

is differentiable wherever $\mathbf{x}^*$ is single-valued with $D_{\boldsymbol{\theta}} v[\theta] = D_{\boldsymbol{\theta}} f[\mathbf{x}(\boldsymbol{\theta}), \boldsymbol{\theta}]$.

Proof. To simplify the proof, assume that $\mathbf{x}^*$ is single-valued for every $\boldsymbol{\theta} \in \Theta$ Then

$$v(\boldsymbol{\theta}) = f(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}) \text{ for every } \boldsymbol{\theta} \in \Theta$$

For any $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0 \in \Theta$

$$\begin{aligned} v(\boldsymbol{\theta}) - v(\boldsymbol{\theta}_0) &= f\big(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}\big) - f\big(\mathbf{x}^*(\boldsymbol{\theta}_0), \boldsymbol{\theta}_0\big) \\ &\geq f\big(\mathbf{x}^*(\boldsymbol{\theta}_0), \boldsymbol{\theta}\big) - f\big(\mathbf{x}^*(\boldsymbol{\theta}_0), \boldsymbol{\theta}_0\big) \\ &= D_{\boldsymbol{\theta}} f[\mathbf{x}^*(\boldsymbol{\theta}_0), \boldsymbol{\theta}_0](\boldsymbol{\theta} - \boldsymbol{\theta}_0) + \eta(\boldsymbol{\theta})\,\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\| \end{aligned}$$

with $\eta(\boldsymbol{\theta}) \to \mathbf{0}$ as $\boldsymbol{\theta} \to \boldsymbol{\theta}_0$. On the other hand, by the mean value theorem (Theorem 4.1) there exist $\bar{\boldsymbol{\theta}} \in (\boldsymbol{\theta}, \boldsymbol{\theta}_0)$ such that

$$\begin{aligned} v(\boldsymbol{\theta}) - v(\boldsymbol{\theta}_0) &= f\big(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}\big) - f\big(\mathbf{x}^*(\boldsymbol{\theta}_0), \boldsymbol{\theta}_0\big) \\ &\leq f\big(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}\big) - f\big(\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}_0\big) \\ &= D_{\boldsymbol{\theta}} f[\mathbf{x}^*(\boldsymbol{\theta}), \bar{\boldsymbol{\theta}}](\boldsymbol{\theta} - \boldsymbol{\theta}_0) \end{aligned}$$

Letting $\boldsymbol{\theta} \to \boldsymbol{\theta}_0$

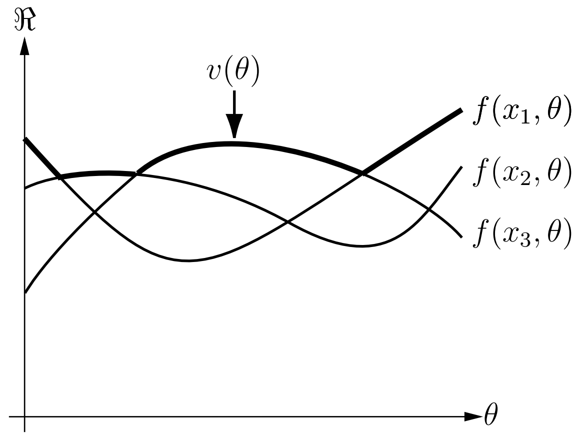
$$\lim_{\boldsymbol{\theta}\to\boldsymbol{\theta}_0} \frac{D_{\boldsymbol{\theta}} f[\mathbf{x}^*(\boldsymbol{\theta}_0), \boldsymbol{\theta}_0](\boldsymbol{\theta} - \boldsymbol{\theta}_0)}{\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|} \leq \lim_{\boldsymbol{\theta}\to\boldsymbol{\theta}_0} \frac{v(\boldsymbol{\theta}) - v(\boldsymbol{\theta}_0)}{\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|} \leq \lim_{\boldsymbol{\theta}\to\boldsymbol{\theta}_0} \frac{D_{\boldsymbol{\theta}} f[\mathbf{x}^*(\boldsymbol{\theta}_0), \boldsymbol{\theta}_0](\boldsymbol{\theta} - \boldsymbol{\theta}_0)}{\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|}$$

v is differentiable (Exercise 4.3) and

$$Dv[\theta] = D_{\boldsymbol{\theta}} f[\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}]$$

where $D_{\boldsymbol{\theta}} f[\mathbf{x}^*(\boldsymbol{\theta}), \boldsymbol{\theta}]$ denotes the partial derivative of f with respect to $\boldsymbol{\theta}$ holding $\mathbf{x}$ constant at $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\theta})$. □

▷ Note that there is no requirement in Theorem 6.2 that f is differentiable with respect to the decision variables $\mathbf{x}$, only with respect to the parameters. The practical importance of dispensing with differentiability with respect to $\mathbf{x}$ is that Theorem 6.2 applies even when the feasible set is discrete (See Example 6.2).



3 Comparative statics of optimization models

There are four different approaches to comparative statics of optimization models

- Revealed preference approach
- Envelope theorem approach
- Monotone maximum theorem approach
- Implicit function theorem approach

3.1 Revealed preference approach

A competitive firm's optimization problem is to choose a feasible production plan $\mathbf{y} \in Y$ to maximize total profit

$$\max_{\mathbf{y} \in Y} \mathbf{p} \cdot \mathbf{y}$$

Consequently, if $\mathbf{y}^1$ maximizes profit when prices are $\mathbf{p}^1$, then

$$\mathbf{p}^1 \cdot \mathbf{y}^1 \geq \mathbf{p} \cdot \mathbf{y} \text{ for every } \mathbf{y} \in Y$$

Similarly, if $\mathbf{y}^2$ maximizes profit when prices are $\mathbf{p}^2$, then

$$\mathbf{p}^2 \cdot \mathbf{y}^2 \geq \mathbf{p} \cdot \mathbf{y} \text{ for every } \mathbf{y} \in Y$$

In particular

$$\mathbf{p}^1 \cdot \mathbf{y}^1 \geq \mathbf{p}^1 \cdot \mathbf{y}^2 \qquad \text{and} \qquad \mathbf{p}^2 \cdot \mathbf{y}^2 \geq \mathbf{p}^2 \cdot \mathbf{y}^1$$

Adding these inequalities

$$\mathbf{p}^1 \cdot \mathbf{y}^1 + \mathbf{p}^2 \cdot \mathbf{y}^2 \geq \mathbf{p}^1 \cdot \mathbf{y}^2 + \mathbf{p}^2 \cdot \mathbf{y}^1$$

Rearranging

$$\mathbf{p}^2 \cdot (\mathbf{y}^2 - \mathbf{y}^1) \geq \mathbf{p}^1 \cdot (\mathbf{y}^2 - \mathbf{y}^1)$$

and therefore

$$(\mathbf{p}^2 - \mathbf{p}^1) \cdot (\mathbf{y}^2 - \mathbf{y}^1) \geq 0 \qquad \text{or} \qquad \sum_{i=1}^n (p_i^1 - p_i^2)(y_i^2 - y_i^2) \geq 0 \tag{6}$$

If prices change from $\mathbf{p}^1$ to $\mathbf{p}^2$, the optimal production plan must change in such a way as to satisfy the inequality (6). For a change in the price of a single good i ($p_j^2 = p_j^1$ for every $j \neq i$), (6) implies that

$$(p_i^2 - p_i^1)(y_i^2 - y_i^1) \geq 0 \qquad \text{or} \qquad p_i^2 > p_i^1 \implies y_i^2 \geq y_i^1$$

3.2 The envelope theorem approach

Letting $f(\mathbf{y}, \mathbf{p}) = \mathbf{p} \cdot \mathbf{y}$ denote the objective function, the competitive firm solves

$$\max_{\mathbf{y} \in Y} f(\mathbf{y}, \mathbf{p})$$

Note that f is differentiable with $D_{\mathbf{p}} f[\mathbf{y}, \mathbf{p}] = \mathbf{y}$. Applying the envelope theorem 6.2, the profit function

$$\Pi(\mathbf{p}) = \sup_{\mathbf{y} \in Y} f(\mathbf{y}, \mathbf{p})$$

is differentiable wherever the supply correspondence $\mathbf{y}^*$ is single-valued with

$$D_{\mathbf{p}} \Pi[\mathbf{p}] = D_{\mathbf{p}} f[\mathbf{y}^*(\mathbf{p}), \mathbf{p}] = \mathbf{y}^*(\mathbf{p}) \tag{7}$$

or

$$\mathbf{y}^*(\mathbf{p}) = \nabla \Pi(\mathbf{p})$$

which is known as **Hotelling's lemma**.

▷ The practical significance of Hotelling's lemma is that, if we know the profit function, we can calculate the supply function by straightforward differentiation instead of solving a constrained optimization problem.

▷ Its theoretical significance is more important. Hotelling's lemma enables us to deduce the properties of the supply function $\mathbf{y}^*$ from the already established properties of the profit function. In particular, we know that the profit function is convex (Example 3.42).

From Hotelling's lemma (7), we deduce that the derivative of the supply function is equal to the second derivative of the profit function

$$D\mathbf{y}^*[\mathbf{p}] = D^2\Pi[\mathbf{p}]$$

or equivalently that the Jacobian of the supply function is equal to the Hessian of the profit function.

$$J_{\mathbf{y}^*}(\mathbf{p}) = H_\Pi(\mathbf{p})$$

Since Π is smooth and convex, its Hessian $H(\mathbf{p})$ is symmetric (Theorem 4.2) and nonnegative definite (Proposition 4.1) for all $\mathbf{p}$. Consequently, the Jacobian of the supply function $J_{\mathbf{y}^*}$ is also symmetric and nonnegative definite. This implies for all goods i and j

$$\begin{aligned} &D_{p_i} y_i^*[\mathbf{p}] \geq 0 && \text{Nonnegativity} \\ &D_{p_i} y_j^*[\mathbf{p}] = D_{p_j} y_i^*[\mathbf{p}] && \text{Symmetry} \end{aligned}$$

In a similar fashion, we can deduce

- Shephard's lemma (Example 6.7)
- Roy's identity (Example 6.8)

From the latter, we can easily derive the Slutsky equation (Example 6.9).

3.3 The implicit function theorem approach

The first-order conditions of an equality constrained optimization problem constitute a system of equations.

$$Q(\mathbf{x}; \boldsymbol{\theta}) = \mathbf{0}$$

Provided the Jacobian ($D_\mathbf{x} Q[\mathbf{x}; \boldsymbol{\theta}]$) of this system is non-singular, we can use the implicit function theorem to solve for $\mathbf{x}^*$ in terms of $\boldsymbol{\theta}$. We illustrate by means of an example.

Example Recall again the chip maker, whose optimization problem is

$$\max_y \theta p y - c(y)$$

The first-order and second-order conditions for profit maximization are

$$Q(y, \theta, p) = \theta p - c'(y) = 0 \text{ and } D_y Q[y, \theta, p] = -c''(y) < 0$$

The second-order condition requires increasing marginal cost. Assuming c is C^2, the first-order condition implicitly defines a function $y(\theta)$. Differentiating the first-order condition with respect to θ, we deduce that

$$p = c''(y) D_{\boldsymbol{\theta}} hetay \qquad \text{or} \qquad D_\theta y = \frac{p}{c''(y)}$$

which is positive by the second-order condition. An increase in yield θ is analogous to an increase in product price p, inducing an increase in output y.

▷ Examples 6.15 and 6.16 apply the same technique to deduce the comparative statics of a competitive multi-input firm.

4 References

- Milgrom, P., and I. Segal (2000), *Envelope Theorems for Arbitrary Choice Sets*. Department of Economics, Stanford University: mimeo.
- Silberberg, E. (1990), *The Structure of Economics: A Mathematical Analysis* (2nd edition). New York, NY: McGraw-Hill.

5 Homework

1. Prove Proposition 5.2, that is if f and $\mathbf{g}$ are C^2 and $Dg[\mathbf{x}^*]$ is of full rank, then the value function
$$v(\mathbf{c}) = \sup\{\ f(\mathbf{x}) : \mathbf{g}(\mathbf{x}) = \mathbf{c}\ \}$$
is differentiable with $\nabla v(\mathbf{c}) = \boldsymbol{\lambda}$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ are the Lagrange multipliers associated with $\mathbf{x}^*$.

2. Suppose that the cost function of a monopolist changes from $c_1(y)$ to $c_2(y)$ in such a way that
$$0 < c_1'(y) < c_2'(y) \text{ for every } \ y > 0$$
Let p_1 denote the profit maximizing price with the cost function $c_1(y)$ and let y_1 be the corresponding output. Similarly let p_2 and y_2 be the profit maximizing price and output when the costs are given by $c_2(y)$.
   (a) Show that
$$c_2(y_1) - c_2(y_2) \geq c_1(y_1) - c_1(y_2) \tag{8}$$
   (b) The "Fundamental Theorem of Calculus" states: *If $f'(x)$ is a continuous function on [a,b], then*
$$f(b) - f(a) = \int_a^b f'(x)dx$$
   Apply this to inequality (8) to deduce that $y_1 \geq y_2$ and therefore that $p_1 \leq p_2$.
   (c) State concisely the proposition you have just proved.

3. Assume that a competitive firm produces a single output y from n inputs $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ according to the production function $y = f(\mathbf{x})$ so as to maximize profit
$$\Pi(\mathbf{w}, p) = \max_{\mathbf{x}} pf(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$$
Assume that there is a unique optimum for every p and $\mathbf{w}$. Show that the input demand $x_i^*(\mathbf{w}, p)$ and supply $y^*(\mathbf{w}, p)$ functions have the following properties:

$D_p y_i^*[\mathbf{w}, p] \geq 0$	Upward sloping supply
$D_{w_i} x_i^*[\mathbf{w}, p] \leq 0$	Downward sloping demand
$D_{w_j} x_i^*[\mathbf{w}, p] = D_{w_i} x_j^*[\mathbf{w}, p]$	Symmetry
$D_p x_i^*[\mathbf{w}, p] = -D_{w_i} y^*[\mathbf{w}, p]$	Reciprocity

Solutions 7

1 The Lagrangean for this problem is

$$L = f(\mathbf{x}) - \boldsymbol{\lambda}^T\big(\mathbf{g}(\mathbf{x}) - \mathbf{c}\big)$$

By Corollary 6.1.1

$$\nabla v(\mathbf{c}) = D_{\mathbf{c}} L = \boldsymbol{\lambda}$$

2 (a) With cost function $c_1(y_1)$, the firms profit is

$$\Pi = py - c_1(y)$$

Since this is maximised at p_1 and y_1 (although the monopolist could have sold y_2 at price p_2)

$$p_1 y_1 - c_1(y_1) \geq p_2 y_2 - c_1(y_2)$$

Rearranging

$$p_1 y_1 - p_2 y_2 \geq c_1(y_1) - c_1(y_2) \tag{1}$$

Similarly

$$p_2 y_2 - c_2(y_2) \geq p_1 y_1 - c_2(y_1)$$

which can be rearranged to yield

$$c_2(y_1) - c_2(y_2) \geq p_1 y_1 - p_2 y_2$$

Combining the previous inequality with (1) yields

$$c_2(y_1) - c_2(y_2) \geq c_1(y_1) - c_1(y_2)$$

(b) Applying the Fundamental Theorem of Calculus to both sides, this implies

$$\int_{y_2}^{y_1} c_2'(y)dy \geq \int_{y_2}^{y_1} c_1'(y)dy$$

or

$$\int_{y_2}^{y_1} c_2'(y)dy - \int_{y_2}^{y_1} c_1'(y)dy = \int_{y_2}^{y_1} (c_2'(y) - c_1'(y))dy \geq 0$$

Since $c_2'(y) - c_1'(y) \geq 0$ for every y (by assumption), this implies that $y_2 \leq y_1$. Assuming the demand curve is downward sloping, this implies $p_2 \geq p_1$.

(c) There is an implicit requirement to utilize the Fundamental Theorem of Calculus, namely that $c'(y)$ is continuous. With this proviso, we have shown that the monopoly price is increasing in marginal cost. Specifically we have shown: Assuming that a monopolist's cost function is continously differentiable (in output), the profit maximizing monopoly price is an increasing (i.e. nondecreasing) function of marginal cost.

3 By Theorem 6.2

$$D_{\mathbf{w}}\Pi[\mathbf{w}, p] = -\mathbf{x}^* \text{ and } D_p\Pi[\mathbf{w}, p] = y^*$$

and therefore

$$\begin{aligned} D_p y(p, \mathbf{w}) &= D^2_{pp}\Pi(p, \mathbf{w}) \geq 0 \\ D_{w_i} x_i(p, \mathbf{w}) &= -D^2_{w_i w_i}\Pi(p, \mathbf{w}) \leq 0 \\ D_{w_j} x_i(p, \mathbf{w}) &= -D^2_{w_i w_j}\Pi(p, \mathbf{w}) = D_{w_i} x_j(p, \mathbf{w}) \\ D_p x_i(p, \mathbf{w}) &= -D^2_{w_i p}\Pi(p, \mathbf{w}) = -D_{w_i} y(p, \mathbf{w}) \end{aligned}$$

since Π is convex and therefore $H_\Pi(\mathbf{w}, p)$ is symmetric (Theorem 4.2) and nonnegative definite (Proposition 4.1).